Workshop and Summer School: From Statistical Mechanics to Conformal and Quantum Field Theory
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Bulk and Boundary Form Factors in QFT (2)

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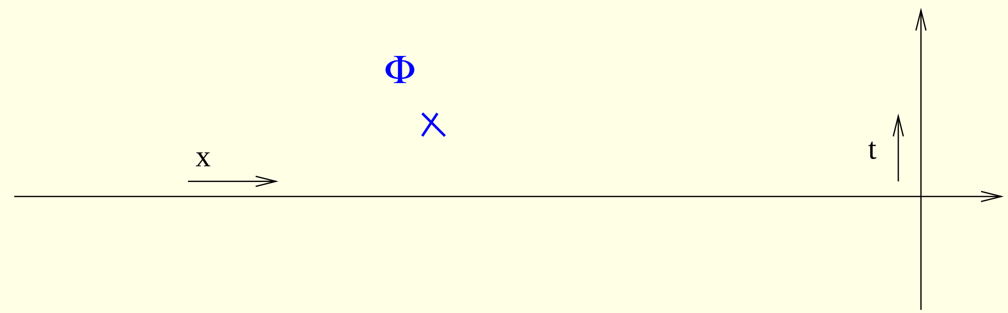
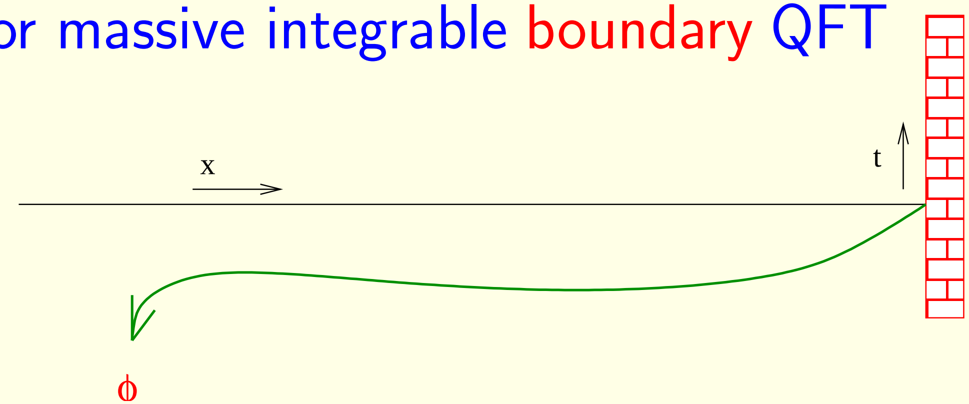
Bulk bootstrap programme for massive integrable QFT in 1+1 D

$$\langle 0|\Phi(x,t)|\theta_1,\theta_2,\dots,\theta_n\rangle^{in}$$



Boundary bootstrap programme for massive integrable boundary QFT

$$_B\langle 0|\varphi_B(t)|\theta_1,\theta_2,\dots,\theta_n\rangle_B^{in}$$



Definition of a QFT: Schemes
QFT={ Correlators of local operators}

$$\mathcal{L} = \frac{1}{2}(\partial_t\Phi)^2 - \frac{1}{2}(\partial_x\Phi)^2 - V(\Phi)$$

Schemes based on a Lagrangian

$\swarrow$

free scheme $V = \frac{1}{2}m^2\Phi^2$

operators: $\mathcal{O} =: \partial^k\Phi \dots \partial^l\Phi :$

solved $\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N)|0\rangle$

$\downarrow$

perturbative scheme $H = H_0 + H_{pert}$

Correlators: $\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N)|0\rangle =$

$$\frac{{}_0\langle 0|T(\mathcal{O}_1 ... \mathcal{O}_N e^{i\int H_{pert}})|0\rangle_0}{{}_0\langle 0|Te^{i\int H_{pert}}|0\rangle_0}$$

$\searrow$

Path integral scheme $(t = -iy)$

operators: $\mathcal{O} = \partial^k\Phi \dots \partial^l\Phi$

Correlators: $\langle \mathcal{O}_1 \dots \mathcal{O}_N\rangle =$

$$\frac{\int \mathcal{D}\Phi\, \mathcal{O}_1 ... \mathcal{O}_N e^{-\int d^2x\, \mathcal{L}_E}}{\int \mathcal{D}\Phi\, e^{-\int d^2x\, \mathcal{L}_E}}$$

Schemes based on symmetries

$\swarrow$

CFT scheme $V = 0$

spectrum from the symmetry

$$H = L_0 + \bar{L}_0 = \frac{1}{2}(\partial_x\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2$$

scaling operators: $\mathcal{O} =: \partial^k\Phi \dots \partial^l\Phi e^{i\beta\Phi} :$

Structure constants: $\langle u|\mathcal{O}|v\rangle$ from consisteny

$$\langle 0|R(\mathcal{O}_1\mathcal{O}_2)|0\rangle = \sum_n \langle 0|\mathcal{O}_1|n\rangle\langle n|\mathcal{O}_2|0\rangle$$

$\searrow$

IFT scheme $V = \cosh\beta\Phi$

spectrum from the symmetry

$$H = \frac{1}{2}(\partial_x\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2 + V(\Phi)$$

S-matrix bootstrap

local operators: $[\mathcal{O}_1, \mathcal{O}_2]_{spl} = 0$

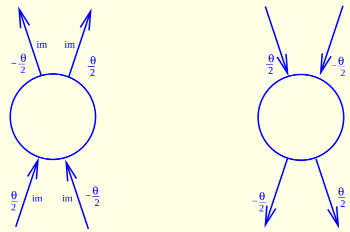
Form factor bootstrap $\langle u|\mathcal{O}|v\rangle$ satiesfies axioms

$$\langle 0|\mathcal{O}_1\mathcal{O}_2|0\rangle = \sum_n \langle 0|\mathcal{O}_1|n\rangle\langle n|\mathcal{O}_2|0\rangle$$

S-matrix bootstrap

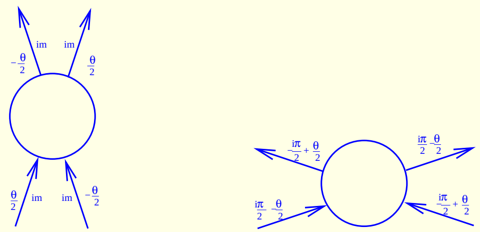
one selfconjugate particle + integrability

Unitarity



$$S(\theta)^{-1} = S(-\theta)$$

Crossing



$$S(\theta) = S(i\pi - \theta)$$

Simplest nontrivial solution $S(\theta) = \frac{\sinh\theta - i\sin\gamma}{\sinh\theta + i\sin\gamma}$

Minimality: all singularity has physical origin: $\gamma > 0$ end of the story (sinh-Gordon):

Bootstrap:

$$S_{new}(\theta) = \qquad S_{old}(\theta + iu) S_{old}(\theta - iu)$$

for $\gamma = -\frac{2}{3}$ self-fusion: Lee-Yang

Analytical properties of the bulk form factors

Bulk form factor $\theta_1 > \theta_2 > \ldots > \theta_n$

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) = \langle 0|\mathcal{O}(0,0)|\theta_1, \ldots, \theta_n\rangle$$

$$F_n^{\mathcal{O}}(\theta_n, \ldots, \theta_1) = \langle 0|\mathcal{O}(0,0)|\theta_n, \ldots, \theta_1\rangle$$

Example $F_2^{\mathcal{O}}(\theta_1, \theta_2) = S(\theta_1 - \theta_2) F_2^{\mathcal{O}}(\theta_2, \theta_1)$

$$F_{mn}^{\mathcal{O}}(\theta_1^{'}, \ldots, \theta_m^{'}|\theta_1, \ldots, \theta_n) = \langle \theta_1^{'}, \ldots, \theta_m^{'}|\mathcal{O}(0,0)|\theta_1, \ldots, \theta_n\rangle$$

Permutation $F_{m2}^{\mathcal{O}}(\theta_1^{'}, \ldots, \theta_m^{'}|\theta_1, \theta_2) = S(\theta_1 - \theta_2) F_{m2}^{\mathcal{O}}(\theta_1^{'}, \ldots, \theta_m^{'}|\theta_2, \theta_1)$

$$< out|\mathcal{O}(x,it)|\theta_1, \theta_2, \ldots, \theta_n > = 2\pi\delta(\theta_1 - \theta)\langle out\backslash\theta|\mathcal{O}(x,it)|\theta_2, \ldots, \theta_n >$$

$$-Z^{-1/2}\int_{-\infty}^{\infty} d(it^{'})\, e^{-i(i\omega(\theta_1))(it^{'})} \int_{-\infty}^{\infty} dx^{'} e^{-ip(\theta_1)x^{'}} \{-\partial_{it^{'}}^2 - \partial_{x^{'}}^2 + m^2\}$$

$$< out|T(\Phi(x^{'}, it^{'})\mathcal{O}(x,it))|\theta_2, \ldots, \theta_n >$$

Crossing from reduction formula

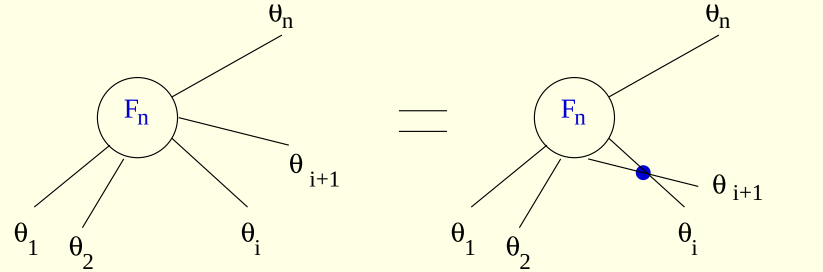
$$F_{1n}^{\mathcal{O}}(\theta|\theta_1, \ldots, \theta_n) = F_{n+1}^{\mathcal{O}}(\theta_1, \ldots, \theta_n, \theta - i\pi) + \delta(\theta - \theta_n) F_{n-1}^{\mathcal{O}}(\theta_1, \ldots, \theta_{n-1})$$

$$F_{1n}^{\mathcal{O}}(\theta|\theta_n, \ldots, \theta_1) = F_{n+1}^{\mathcal{O}}(\theta + i\pi, \theta_n, \ldots, \theta_1) + \delta(\theta - \theta_n) F_{n-1}^{\mathcal{O}}(\theta_{n-1}, \ldots, \theta_1)$$

Bulk form factor axioms

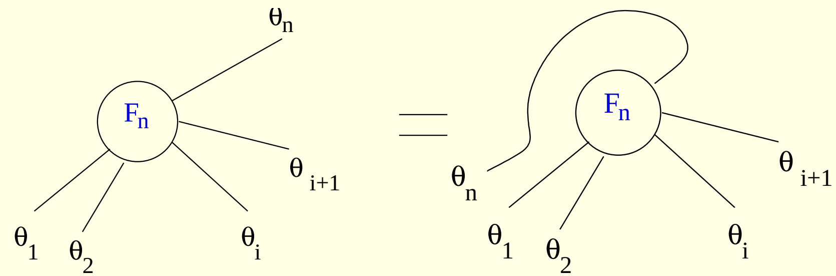
Permutation

$$F_n^{\mathcal{O}}(\theta_1,\ldots\theta_i,\theta_{i+1},\ldots,\theta_n)=$$
$$S(\theta_i-\theta_{i+1})F_n^{\mathcal{O}}(\theta_1,\ldots\theta_{i+1},\theta_i,\ldots,\theta_n)$$



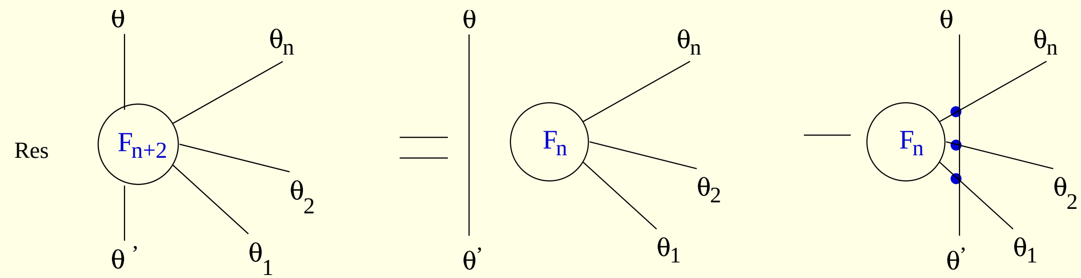
Periodicity

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{n-1},\theta_n)=$$
$$F_n^{\mathcal{O}}(\theta_n+2\pi i,\theta_1,\ldots,\theta_{n-1})$$



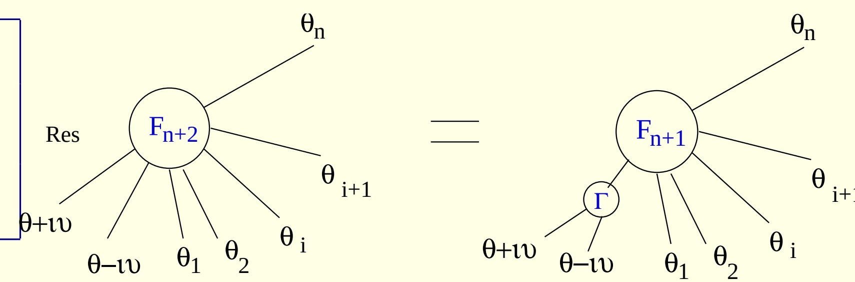
Kinematical singularities

$$-i\mathrm{res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta+i\pi,\theta',\theta_1,\ldots,\theta_n)=$$
$$=\left(1-\prod_{i=1}^{n}S(\theta-\theta_i)\right)F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$



Dynamical singularities

$$-i\mathrm{res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta+iu,\theta'-iu,\theta_1,\ldots,\theta_n)=$$
$$\Gamma F_{n+1}^{\mathcal{O}}(\theta,\theta_1,\ldots,\theta_n)$$



Solving the bulk form factor equations

Bulk theory with $S(\theta)$: FF axioms + minimality $\longrightarrow$ Solution of the FF equations

Step 1. Solve first the two particle case $F_2(\theta_1 - \theta_2) = f(\theta)$

$$f(\theta) = S(\theta) f(-\theta) \quad ; \quad f(i\pi + \theta) = f(i\pi - \theta)$$

minimality: only dynamical poles, one zero at $\theta = 0$, minimal growth at $\theta \to \infty$

Step 2. Take the Ansatz to satisfy the permutation and the periodicity axioms

$$F_n(\theta_1, \dots, \theta_n) = H_n Q_n(x_1, \dots, x_n) \prod_{i<j} f(\theta_i - \theta_j) \frac{1}{x_i + x_j} \quad ; \quad x = e^\theta$$

$Q_n(x_1, \dots, x_n)$: completely symmetric polynomial $\leftrightarrow$ locality

Step 3. Derive recursion relations from the singularity axioms

kinematical $Q_{n+2}(-x, x, x_1, \dots, x_n) = P_n(x|x_1, \dots, x_n) Q_n(x_1, \dots, x_n)$

dynamical $Q_{n+1}(e^{iu}x, e^{-iu}x, x_1, \dots, x_n) = R_n(x|x_1, \dots, x_n) Q_n(x, x_1, \dots, x_n)$

Step 4. Solve recursion, compute 2ptf

$$\langle 0|\mathcal{O}(x,0)\mathcal{O}(0,0)|0\rangle = \sum_{n=0}^{\infty} (-1)^n \int_0^\infty \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} |F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n)|^2 e^{-mx \sum_{i=1}^n \cosh\theta_i}$$

Perturbed Lee-Yang model

Bulk theory with $S(\theta)=\frac{\sinh\theta+i\sin\frac{\pi}{3}}{\sinh\theta-i\sin\frac{\pi}{3}}$

Step 1. Solution of the two particle case $F_2(\theta_1-\theta_2)=f(\theta)$

$$f(\theta)=\frac{\cosh\theta-1}{\cosh\theta+\frac{1}{2}}v(i\pi-\theta)v(-i\pi+\theta)\quad ; \quad v(\theta)=\exp\left\{2\int_0^\infty\frac{dx}{x}e^{i\theta x/\pi}\frac{\sinh\frac{x}{2}\sinh\frac{x}{3}\sinh\frac{x}{6}}{\sinh^2 x}\right\}$$

minimality: dynamical pole at $\theta=\frac{2i\pi}{3}$, zero at $\theta=0$, growth: $f(\theta)\to 1$ for $\theta\to\infty$

Step 2. The Ansatz satisfying the permutation and periodicity axioms

$$F_n(\theta_1,\dots,\theta_n)=H_nQ_n(x_1,\dots,x_n)\prod_{i<j}\frac{f(\theta_i-\theta_j)}{x_i+x_j}\quad ; \qquad H_n=<\mathcal{O}>\left(\frac{3^{1/4}}{2^{1/2}v(0)}\right)^n$$

$Q_n(x_1,\dots,x_n)$: completely symmetric polynomial

Step 3. Recursion relations from the singularity axioms

kinematical $Q_{n+2}(-x,x,x_1,\dots,x_n)=P_n(x|x_1,\dots,x_n)Q_n(x_1,\dots,x_n)$

$$P_n(x|x_1,\dots,x_n)=\frac{(-1)^nx^2}{2(x_+-x_-)}\left(\prod_{i=1}^n(x_i+x_+)(x_i-x_-)-\prod_{i=1}^n(x_i-x_+)(x_i+x_-)\right)\quad x_\pm=e^{\mp i\frac{\pi}{3}}x$$

dynamical $Q_{n+1}(x_+,x_-,x_1,\dots,x_n)=x\prod_{i=1}^n(x+x_i)Q_n(x,x_1,\dots,x_n)$

Perturbed bulk Lee-Yang: minimal form factor solution

Step 4. Solve recursion, compute 2pt function

$Q_1 = 1; \quad Q_2 = \sigma_1; \quad Q_3 = \sigma_1\sigma_2 \dots ; \quad Q_n = det(\sigma_{2i-j}[i-j+1])$

$\prod_{i=1}^{n}(x+x_i) = \sum_{k=0}^{n} x^k \sigma_{n-k}$

The solution with minimal degree is unique. Which operator is it? Compute 2pt:

$\langle 0|\mathcal{O}(x,0)\mathcal{O}(0,0)|0\rangle = \sum_{n=0}^{\infty}(-1)^n \int_0^{\infty} \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} |F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n)|^2 e^{-mx\sum_{i=1}^{n}\cosh\theta_i}$

Large distance expansion $\leftrightarrow$ Compare to short distance (UV) expansion

$$\mathcal{A} = \mathcal{A}_{(2,5)} + g\int d^2x \Phi_{(-\frac{1}{5},-\frac{1}{5})}$$

$\Phi_{(-\frac{1}{5},-\frac{1}{5})} = \Phi$ field with smallest scaling dimension

$\langle 0|\Phi(x,0)\Phi(0,0)|0\rangle = x^{\frac{4}{5}} + x^{\frac{2}{5}} C_{\Phi\Phi}^{\Phi}\langle\Phi\rangle + \dots$

Perturbed bulk Lee-Yang two point function

Conformal limit compared to form factor expansion

$$\langle 0|\Phi(x,0)\Phi(0,0)|0\rangle$$

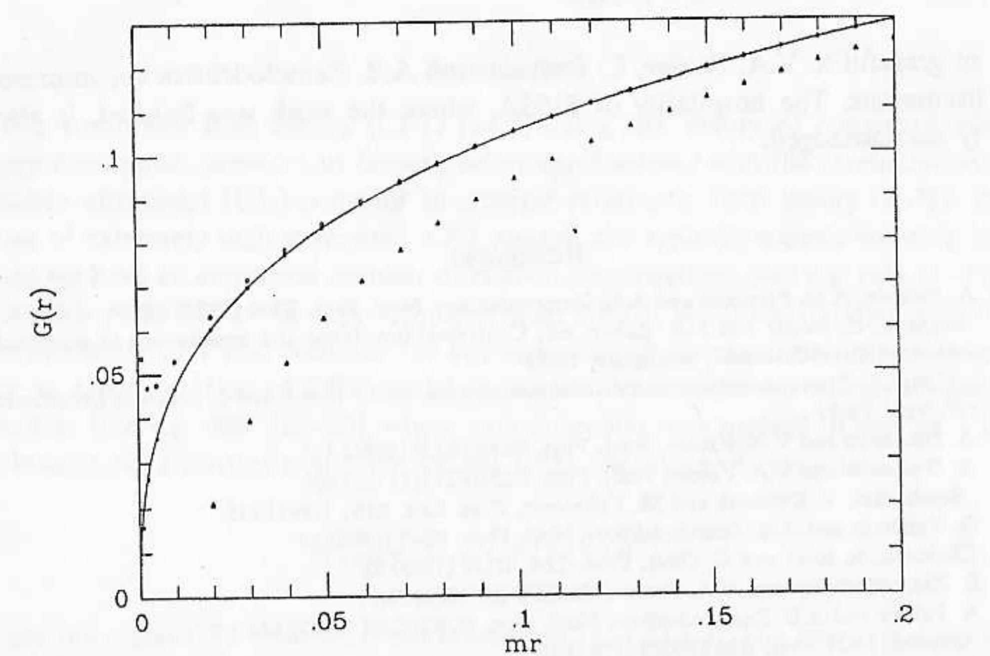


Fig. 3. Convergence of the large-distance expansion for small mr. Empty triangles: zero- and one-particle contributions. Empty circles: the same plus two-particle term. Full circles: up to three-particle state contributions. Full curve: the short-distance data.

Conformal limit: $x^{\frac{4}{5}} + x^{\frac{2}{5}} C_{\Phi\Phi}^{\Phi} \langle \Phi \rangle$

FF expansion 0+1+2+3pt: $|F_0^{\Phi}|^2 +$

$$-\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} |F_1^{\Phi}|^2 e^{-mx\cosh\theta}$$

$$+\int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{2(2\pi)^2} |F_2^{\Phi}(\theta_1,\theta_2)|^2 \cdot e^{-mx(\cosh\theta_1+\cosh\theta_2)}$$

$$+\int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2 d\theta_3}{6(2\pi)^3} |F_3^{\Phi}(\theta_1,\theta_2,\theta_3)|^2 \dots$$

Operator classification: visit Aldo's talk

FF solutions $+$ asymptotics (Q is a polynomial) $\longrightarrow$ locality

FF solutions $\leftrightarrow$local operator content

Problem: identification of the operators $\longrightarrow$ compare with other schemes,

Example: derivative operators

$$\langle 0|\Phi(x,t)|\theta_1,\ldots,\theta_N\rangle = e^{i\sum_i \cosh\theta_i t - i\sum_i \sinh\theta_i x} F(\theta_1,\ldots,\theta_N)$$

$\partial_\pm = \partial_t \pm \partial_x$ corresponds to multiplication by σ_1 or $\frac{\sigma_{n-1}}{\sigma_n}$

perturbed free: Energy momentum tensor, spin, conservation laws, equation of motion

pertubed conformal, Δ theorem, clustering, spin

Generalization

Solve the form factor equations for $\gamma > 0$: sinh-Gordon model

solve other models with one selfconjugate particle: Bullough-Dodd

Extend the FF axioms for more particles with diagonal scattering

Extend the FF axioms for more particles with nondiagonal scattering

Use FFs to compute a perturbation of an integrable model

Try to sum up the FF series

Try to describe FFs in finite volume

Generalize the FF program to integrable boundary QFTs

History of form factors

M. Karowski et al Nucl.Phys.B139:455,1978, Phys.Rept.49:229-237,1979, FF program without periodicity, explicite solutions upto two particles, Phys.Rev.D19:2477,1979, summing up the FFs in the Ising case

F. Smirnov et al, Nucl.Phys.B337:156-180,1990. solution of Lee-Yang, restricted sG, Advanced Series in Mathematical physics Vol. 14: the nondiagonal FF program, with solutions to s-G, O(3) σ, $SU(N)$ Thirring, locality is proved.

Mussardo et al Nucl.Phys. B393 (1993) 413-441, Phys.Lett. B307 (1993) 83-90, Phys.Lett. B311 (1993), Phys.Lett. B317 (1993) 573-580, Int. J. Mod. Phys. A9 (1994) 3307-3338, Phys.Lett. B324 (1994) 40-44, FF for sh-G, Bullough-Dodd, staircase, $\mathcal{M}_{3,5}+\Phi_{1,3}$... on FF perturbation theory see Nucl.Phys.B473:469-508,1996, Nucl.Phys.B516:675-703,1998., Nucl.Phys.B737:291-303,2006.

Operator identifications: ask Aldo

Finite volume FF: ask Giuseppe or Aldo

And many other works as well ...

Coming soon: a book about FFs, by O. Castro-Alvaredo, A. Fring, M. Karowski