From Statistical Mechanics to Conformal and Quantum Field Theories

University of Melbourne

8 January–8 February 2007

FROM PARTICLES TO OPERATORS IN INTEGRABLE QFT

Gesualdo Delfino

SISSA-Trieste

QFT is a very successful tool of theoretical physics, wiht applications ranging from **particle physics** to **condensed matter** and **critical phenomena**

Same tool for very different observables:

particles (S-matrix) for particle physics

operators (correlation functions) for critical phenomena

The **equivalence of the particle and operator descriptions** is far from obvious for an interacting QFT

(the dominant line of thought in the '50s–'60s was that the concept of field (operator) should be banished from particle physics)

Today we believe that the knowledge of all correlation functions **or** the knowledge of the full S-matrix separately amount to the **complete solution** of a QFT

From operators to particles

The **reduction formula** allows to compute the S-matrix from the correlation functions of the operators which interpolate the particles

Notice that

– not all the operators interpolate particles

– there are infinitely many interpolating operators for the same particle

– in the reduction procedure (off-shell $\rightarrow$ on-shell) we loose information

$\Longrightarrow$ It is impossible to go the opposite way (from the S-matrix to the correlators) in this framework

From particles to operators

Is is believed that, if the S-matrix is known, the matrix elements of the operators on asymptotic particle states (**form factors**) can be computed through a **bootstrap** procedure based on analytic properties

Correlation functions are then obtained as spectral sums over the form factors

In principle, this form factor program

– should account for the infinitely many operators of the theory

– should apply to any theory, including the strongly interacting ones for which the operator content is not known a priori

In practice, in the general case we do not even know the starting point, i.e. the S-matrix, and the details of the form factor program remain unclear

The integrable theories in $d = 2$ provide up to now the only place in QFT where this important theoretical issue can be tested non-perturbatively

Integrable massive theories

The existence of an infinite number of conserved quantities allows the determination of the **exact S-matrix**, which is completely elastic and factorizable. Since these theories describe the scaling limit of statistical models, obtaining the correlation functions from the S-matrix is not only important in principle, but also necessary for the physical applications

Form factor equations (Karowski, Weisz, '78; Smirnov, '80s)

$F_n^\Phi(\theta_1,\ldots,\theta_n)=\langle 0|\Phi(0)|\theta_1\ldots\theta_n\rangle$ $\qquad$ $(p_i^0,p_i^1)=(m\cosh\theta_i,m\sinh\theta_i)$

$$i)\qquad F_n^\Phi(\theta_1+\alpha,\ldots,\theta_n+\alpha)=e^{s_\Phi\alpha}F_n^\Phi(\theta_1,\ldots,\theta_n)$$

$$ii)\qquad F_n^\Phi(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n)=S(\theta_i-\theta_{i+1})\,F_n^\Phi(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n)$$

$$iii)\qquad F_n^\Phi(\theta_1+2i\pi,\theta_2,\ldots,\theta_n)=F_n^\Phi(\theta_2,\ldots,\theta_n,\theta_1)$$

$$iv)\qquad \text{Res}_{\theta'=\theta}\,F_{n+2}^\Phi(\theta'+iu/2,\theta-iu/2,\theta_1,\ldots,\theta_n)=i\Gamma\,F_{n+1}^\Phi(\theta,\theta_1,\ldots,\theta_n)$$

$$v)\qquad \text{Res}_{\theta'=\theta+i\pi}\,F_{n+2}^\Phi(\theta',\theta,\theta_1,\ldots,\theta_n)=i[1-\prod_{j=1}^{n}S(\theta-\theta_j)]F_n^\Phi(\theta_1,\ldots,\theta_n)$$

The essential simplification coming from integrability is in the unitarity equation $ii)$

scattering data: $S(\theta)$, u, Γ

operator data: s_Φ + internal symmetries (if any)

The solutions form a linear space that must be infinite-dimensional

We know what structure to expect for the space of solutions

Operator space in 2d QFT

- **Conformal field theories** (BPZ, '84)

Correspondence with the highest weight representations of the Virasoro algebra. The operator space splits into **operator families**, each one consisting of a **primary** Φ_0 and infinitely many **descendants**

$$\Phi(x) = L_{-i_1} \dots L_{-i_I} \bar{L}_{-j_1} \dots \bar{L}_{-j_J} \, \Phi_0(x)$$

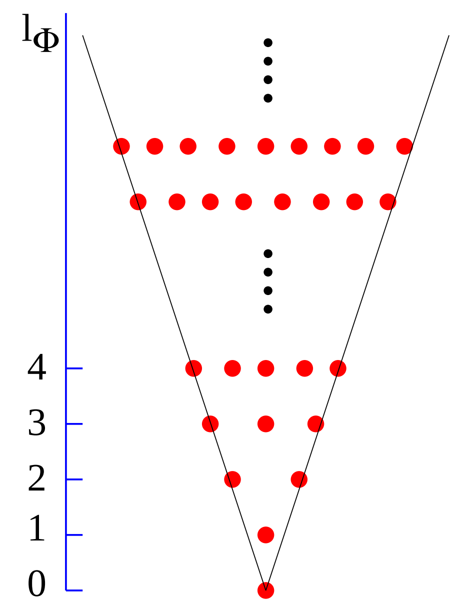
$$0 < i_1 \le i_2 \le \dots \le i_I \qquad 0 < j_1 \le j_2 \le \dots \le j_J$$

$$(l_\Phi, \bar{l}_\Phi) = \left(\sum_{n=1}^{I} i_n \, , \sum_{n=1}^{J} j_n \right) \qquad \text{left and right level}$$

$$(\Delta_\Phi, \bar{\Delta}_\Phi) = (\Delta_{\Phi_0} + l_\Phi, \bar{\Delta}_{\Phi_0} + \bar{l}_\Phi) \qquad \text{conformal dimensions}$$

$$X_\Phi = \Delta_\Phi + \bar{\Delta}_\Phi \qquad \text{scaling dimension}$$

$$s_\Phi = \Delta_\Phi - \bar{\Delta}_\Phi \qquad \text{spin}$$



The actual 'filling' of each level depends on the existence of null vectors

- **Massive theories**

Can be seen as perturbations of the conformal theories.

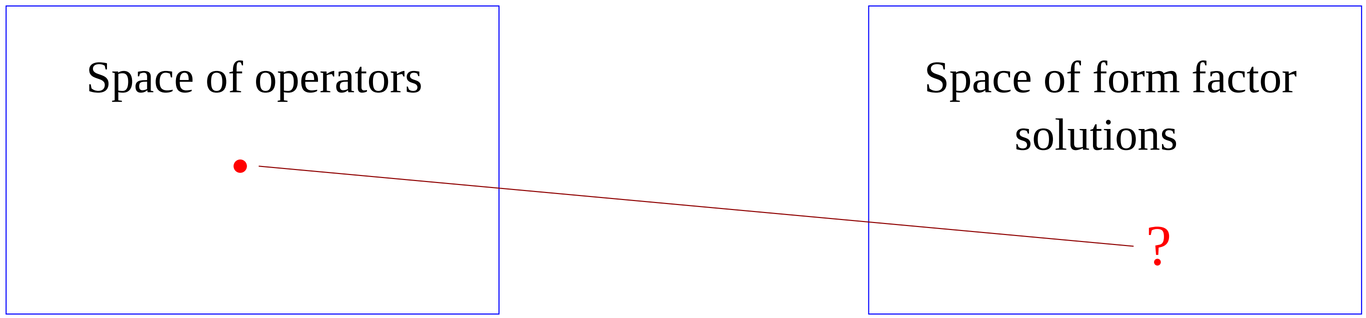
Conformal perturbation theory indicates that **the operator space is isomorphic to the conformal one** (A.Zamolodchikov, '88; Al.Zamolodchikov, '91; Guida, Magnoli, '95)

Test of the isomorphism (Cardy, Mussardo, '90) : count the number of solution of the form factor equations for each value of the spin s_Φ and compare with the conformal characters.

Successfully checked for several models (Cardy, Mussardo, '90; Koubek, '95; Smirnov, '95; Babelon, Bernard, Smirnov, '97; Jimbo, Miwa, Takeyama, '03)

Beyond counting, what is the **correspondence between operators and solutions of the form factor equations ?**

Identification problem



The form factor equations do not contain enough information about the operator to establish the one-to-one correspondence

For long time the working tool has been the **minimality assumption**: the ‘minimal’ form factor solution in a given sector of internal symmetry corresponds to the primary operator with that symmetry

What if there is not enough symmetry? What about descendants?

Asymptotic constraints

high energy limit $\sim$ conformal limit

- Bound in reflection positive theories (GD, Mussardo, '95)

$$\lim_{|\theta_i|\to\infty} F_n^{\Phi}(\theta_1,\ldots,\theta_n) \sim \exp(y_\Phi|\theta_i|) \qquad y_\Phi \leq X_\Phi/2 \qquad (s_\Phi = 0)$$

- Factorization for primaries (GD, Simonetti, Cardy, '96)

$$\lim_{\lambda\to+\infty} F_n^{\Phi_0}(\theta_1+\lambda,\ldots,\theta_k+\lambda,\theta_{k+1},\ldots,\theta_n) = \frac{1}{\langle\Phi_0\rangle} F_k^{\Phi_0}(\theta_1,\ldots,\theta_k) F_{n-k}^{\Phi_0}(\theta_{k+1},\ldots,\theta_n)$$

non-linear in Φ_0 $\qquad (F_n^{\Phi_0} \neq 0\ \forall n)$

Sum rule for scaling dimensions (GD, Simonetti, Cardy, '96)

$$X_{\Phi_0} = -\frac{2}{\pi\langle\Phi_0\rangle}\int d^2x\,\langle\Theta(x)\Phi_0(0)\rangle_{conn} \qquad \Theta = \frac{\pi}{2}T_\mu^\mu$$

Example: primaries in the Ising model

- Thermal case (Berg, Karowski, Weisz, '79)

  σ and ϵ are distinguished by the spin reversal symmetry, minimality works

- Magnetic integrable case (GD, Mussardo, '95; GD, Simonetti, '96)

  no internal symmetry, asymptotic factorization is essential

	Field theory	Numerical[1]
$F_1^\sigma/\langle\sigma\rangle$	−0.640902..	−0.6408(3)
$F_1^\varepsilon/\langle\varepsilon\rangle$	−3.70658..	−3.707(7)

[1] Caselle, Hasenbusch, '00 (transfer matrix on the lattice)

Asymptotics for descendants (GD, Niccoli, '04)

The minimal form factor solutions are actually the solutions with the mildest asymptotic behavior. Each one corresponds to a primary Φ_0.

The descendants of Φ_0 are among the infinitely many solutions F_n^{Φ} such that

$$\lim_{\theta_i \to \pm\infty} \frac{F_n^{\Phi}(\theta_1, \ldots, \theta_n)}{F_n^{\Phi_0}(\theta_1, \ldots, \theta_n)} \sim \exp(\pm N_\Phi^\pm \theta_i) \qquad N_\Phi^\pm = 1, 2, \ldots$$

Compare with $\Delta_\Phi - \Delta_{\Phi_0} = l_\Phi$

Assume that the form factors of descendants of the same level have the same asymptotic behavior. Then

$$N_\Phi^+ = l_\Phi \qquad N_\Phi^- = \bar{l}_\Phi$$

since $L_{-1} = \partial$ $\bar{L}_{-1} = \bar{\partial}$ and

$$F_n^{\partial^l \bar{\partial}^{\bar{l}} \Phi_0}(\theta_1, \ldots, \theta_n) \propto \left(\sum_{j=1}^n e^{\theta_j} \right)^l \left(\sum_{j=1}^n e^{-\theta_j} \right)^{\bar{l}} F_n^{\Phi_0}(\theta_1, \ldots, \theta_n)$$

In this way form factor solutions can be classified according to the level

The asymptotic conditions take their simplest form when $F_n^\Phi \neq 0 \ \forall n$

$$\lim_{\theta_i \to +\infty} F_n^\Phi(\theta_1, \ldots, \theta_n) \sim \exp(l_\Phi \theta_i) \qquad\qquad (n > 1, \Phi \neq \text{chiral descendant of } I)$$

Consider $\quad \Phi = \mathcal{L}_l \bar{\mathcal{L}}_{\bar{l}} \Phi_0 \quad$ with

$$\mathcal{L}_l \ \longrightarrow \ L_{-i_1} \ldots L_{-i_I} \qquad \bar{\mathcal{L}}_{\bar{l}} \ \longrightarrow \ \bar{L}_{-j_1} \ldots \bar{L}_{-j_J} \qquad \text{in the critical limit}$$

$$l = \sum_{n=1}^{I} i_n \qquad \bar{l} = \sum_{n=1}^{J} j_n$$

Then

$$\lim_{\lambda \to +\infty} e^{-l\lambda} F_n^{\mathcal{L}_l \bar{\mathcal{L}}_{\bar{l}} \Phi_0}(\theta_1 + \lambda, .., \theta_k + \lambda, \theta_{k+1}, .., \theta_n) = \frac{1}{\langle \Phi_0 \rangle} F_k^{\mathcal{L}_l \Phi_0}(\theta_1, .., \theta_k) \, F_{n-k}^{\bar{\mathcal{L}}_{\bar{l}} \Phi_0}(\theta_{k+1}, .., \theta_n)$$

generalizes asymptotic factorization to descendants

Descendants from conserved quantities

An integrable theory possesses conservation laws

$$\bar{\partial} T_{s+1} = \partial \Theta_{s-1} \qquad \text{for an infinite set of positive integers } s$$

$$Q_s = \int_{-\infty}^{+\infty} dx_1 \, [T_{s+1} + \Theta_{s-1}] \qquad \text{are conserved}$$

(similarly there exist $\bar{Q}_s$ with spin $-s$ $\qquad Q_1 = P^0 + P^1 \qquad \bar{Q}_1 = P^0 - P^1$)

$[Q_s, \Phi]$ with dimensions $(\Delta_\Phi + s, \bar{\Delta}_\Phi)$ gives the variation of Φ under the transformation generated by Q_s

$$Q_s|0\rangle = 0 \qquad Q_s|\theta_1, \ldots, \theta_n\rangle = \Lambda_n^{(s)}(\theta_1, .., \theta_n)\, |\theta_1, \ldots, \theta_n\rangle$$

$$\Lambda_n^{(s)}(\theta_1, .., \theta_n) = \kappa_s \, m^s \sum_{i=1}^{n} e^{s\theta_i}$$

$$F_n^{[Q_s,\Phi]}(\theta_1, \ldots, \theta_n) = -\Lambda_n^{(s)}(\theta_1, .., \theta_n)\, F_n^{\Phi}(\theta_1, \ldots, \theta_n)$$

satisfy the form factor equations for spin $s_\Phi + s$

i) $e^{s\theta} + e^{s(\theta + i\pi)} = 0 \quad \Longrightarrow \quad s$ odd

ii) if $\Gamma \neq 0$: $\qquad e^{is\pi/3} + e^{-is\pi/3} = 1 \quad \Longrightarrow \quad s = 6n \pm 1$

Not all descendants can be generated using the conserved quantities

Descendants in the massive Lee-Yang model

(GD, Niccoli, '05)

- **Conformal point** Simplest non-trivial fixed point. Minimal model $\mathcal{M}_{2,5}$ with two local primaries

$I = \phi_{1,1} = \phi_{1,4}$ with $(\Delta, \bar{\Delta}) = (0, 0)$

$\varphi = \phi_{1,2} = \phi_{1,3}$ with $(\Delta, \bar{\Delta}) = (-1/5, -1/5)$

Characters: $\chi_\phi(q) = \sum_{n=0}^{\infty} d_\phi(n) q^n$

$d_\phi(l) d_\phi(\bar{l})$ = number of descendants of ϕ at level $(l, \bar{l})$

$$\chi_I(q) = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + O(q^9)$$

$$\chi_\varphi(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + O(q^9)$$

$(l, \bar{l})$	I	φ
$(1,0)$	0	$\partial\varphi$
$(2,0)$	$T = L_{-2} I$	$\partial^2\varphi$
$(3,0)$	∂T	$\partial^3\varphi$
$(4,0)$	$\partial^2 T$	$\partial^4\varphi\,;\ L_{-4}\varphi$
$(5,0)$	$\partial^3 T$	$\partial^5\varphi\,;\ \partial L_{-4}\varphi$
$(6,0)$	$\partial^4 T\,;\ L_{-4} T$	$\partial^6\varphi\,;\ \partial^2 L_{-4}\varphi\,;\ L_{-6}\varphi$
$(7,0)$	$\partial^5 T\,;\ \partial L_{-4} T$	$\partial^7\varphi\,;\ \partial^3 L_{-4}\varphi\,;\ \partial L_{-6}\varphi$

- **Massive integrable theory** $\mathcal{A} = \mathcal{A}_{\mathcal{M}_{2,5}} - ig \int d^2x\, \varphi(x)$

Conserved quantities Q_s with $s = 6n \pm 1$ A single particle A with $AA \to A$

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2i\pi}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2i\pi}{3}\right)}$$ (Cardy, Mussardo, '89)

Notation: $[Q_s, \Phi] \longrightarrow Q_s\Phi$

$(l, \bar{l})$	I	φ
$(1,0)$	0	$\partial\varphi$
$(2,0)$	T	$\partial^2\varphi$
$(3,0)$	∂T	$\partial^3\varphi$
$(4,0)$	$\partial^2 T$	$\partial^4\varphi$; $R_4\varphi$
$(5,0)$	$\partial^3 T$	$\partial^5\varphi$; $Q_5\varphi$
$(6,0)$	$\partial^4 T$; $S_4 T$	$\partial^6\varphi$; $\partial Q_5\varphi$; $R_6\varphi$
$(7,0)$	$\partial^5 T$; $Q_5 T$	$\partial^7\varphi$; $\partial^2 Q_5\varphi$; $Q_7\varphi$

$Q_5 T = \partial S_4 T$ $Q_5\varphi = \partial R_4\varphi$ $Q_7\varphi = \partial R_6\varphi$

$S_4 \sim \partial^4 + aL_{-4}$ $R_4 \sim \partial^4 + bL_{-4}$ $R_6 \sim \partial^6 + c\partial^2 L_{-4} + dL_{-6}$ $a, b, d \neq 0$

Indeed $Q_5 T \sim \partial^5 T + a\partial L_{-4} T$ with $a \neq 0$, etc

Form factors

- **Primaries** $F_n^I(\theta_1,\ldots,\theta_n) = \delta_{n,0}$

  F_n^{φ} are known (Smirnov, '90; Al.Zamolodchikov, '91)

- **Descendants of φ**

$$F_n^{Q_l\bar{Q}_{\bar{l}}\varphi}(\theta_1,\ldots,\theta_n) = \Lambda_n^{(l)}(\theta_1,..,\theta_n)\,\bar{\Lambda}_n^{(\bar{l})}(\theta_1,..,\theta_n)\,F_n^{\varphi}(\theta_1,\ldots,\theta_n)$$

Up to level 7, we are left with the operators obtained applying R_k and $\bar{R}_k$ $(k=4,,6)$ to φ

Example: $Q_5\varphi = \partial R_4\varphi \quad \Longrightarrow \quad F_n^{R_4\varphi} = \left(i\,m\,\sigma_1^{(n)}\right)^{-1} F_n^{Q_5\varphi}$

$\sigma_1^{(n)} = x_1+\ldots+x_n \qquad x_i = e^{\theta_i} \qquad \prod_{i=1}^n (x+x_i) = \sum_{k=0}^n x^{n-k}\sigma_k^{(n)}(x_1,\ldots,x_n)$

$\left(i\,m\,\sigma_1^{(n)}\right)^{-1}$ does not introduce poles in $F_n^{R_4\varphi}$ because $\varphi\sim\Theta$ Indeed:

$\bar{\partial}T = \partial\Theta \qquad \partial\bar{T} = \bar{\partial}\Theta \qquad \Longrightarrow \qquad F_n^T = -\frac{\sigma_n^{(n)}\sigma_1^{(n)}}{\sigma_{n-1}^{(n)}}F_n^{\Theta} \qquad F_n^{\bar{T}} = -\frac{\sigma_{n-1}^{(n)}}{\sigma_1^{(n)}\sigma_n^{(n)}}F_n^{\Theta}$

$$\Longrightarrow \qquad F_n^{\Theta} \propto \sigma_1^{(n)}\sigma_{n-1}^{(n)}$$

- **Descendants of I**

All chiral descendants of I up to level 7 can be expressed in terms of the form factors of T and $\bar{T}$

The lowest non-chiral descendant of I is $T\bar{T}$ (discussed later)

All non-chiral descendants of I up to level 7 can be expressed in terms of the form factors of $T\bar{T}$

Summary

– All operators of the massive Lee–Yang model up to level 7 can be expressed in terms of the form factors of $\Theta \sim \varphi$, T, $\bar{T}$ and $T\bar{T}$

– These solutions automatically satisfy the asymptotic properties

$$\lim_{\theta_i \to +\infty} F_n^{\Phi}(\theta_1, \ldots, \theta_n) \sim \exp(l_\Phi \theta_i) \qquad (n > 1, \Phi \neq \text{chiral descendant of } I)$$

$$\lim_{\lambda \to +\infty} e^{-l\lambda} F_n^{\mathcal{L}_l \bar{\mathcal{L}}_{\bar{l}} \Phi_0}(\theta_1 + \lambda, .., \theta_k + \lambda, \theta_{k+1}, .., \theta_n) = \frac{1}{\langle \Phi_0 \rangle} F_k^{\mathcal{L}_l \Phi_0}(\theta_1, .., \theta_k)\, F_{n-k}^{\bar{\mathcal{L}}_{\bar{l}} \Phi_0}(\theta_{k+1}, .., \theta_n)$$

– Beyond level 7 not all operators can be related to the conserved quantities

Comparison with the form factor boostrap

Scalar solutions of the form factor equations $(D_n \equiv \sigma_1^{(n)}\sigma_{n-1}^{(n)}/\sigma_n^{(n)})$

(0,0)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	(6,6)	(7,7)
$F_n^{\ominus}$	$D_n F_n^{\ominus}$	$(D_n)^2 F_n^{\ominus}$	$(D_n)^3 F_n^{\ominus}$	$(D_n)^4 F_n^{\ominus}$	$(D_n)^5 F_n^{\ominus}$	$(D_n)^6 F_n^{\ominus}$	$(D_n)^7 F_n^{\ominus}$
F_n^I		$F_n^{K_3}$	$D_n F_n^{K_3}$	$(D_n)^2 F_n^{K_3}$	$(D_n)^3 F_n^{K_3}$	$(D_n)^4 F_n^{K_3}$	$(D_n)^5 F_n^{K_3}$
				F_n^{A,K_3}	$D_n F_n^{A,K_3}$	$(D_n)^2 F_n^{A,K_3}$	$(D_n)^3 F_n^{A,K_3}$
				F_n^{B,K_3}	$D_n F_n^{B,K_3}$	$(D_n)^2 F_n^{B,K_3}$	$(D_n)^3 F_n^{B,K_3}$
						F_n^{C,K_3}	$D_n F_n^{C,K_3}$
						F_n^{D,K_3}	$D_n F_n^{D,K_3}$
				$F_n^{K_4}$	$D_n F_n^{K_4}$	$(D_n)^2 F_n^{K_4}$	$(D_n)^3 F_n^{K_4}$
						F_n^{A,K_4}	$D_n F_n^{A,K_4}$
						F_n^{B,K_4}	$D_n F_n^{B,K_4}$
						F_n^{C,K_4}	$D_n F_n^{C,K_4}$
						F_n^{D,K_4}	$D_n F_n^{D,K_4}$
						F_n^{E,K_4}	$D_n F_n^{E,K_4}$
						$F_n^{K_5}$	$D_n F_n^{K_5}$
2	1	2	2	5	5	13	13

The classification according to the level uses the asymptotic conditions

$F_n^{K_M}(\theta_1,\ldots,\theta_n) = 0$ $\qquad n = 0,1,\ldots,M-1$ $\qquad$ kernel solution

This basis and that obtained from conserved quantities span the same space

The operator $T\bar{T}$

A.Zamolodchikov ('04) showed that this composite operator can be defined as

$$T\bar{T}(x) = \lim_{\epsilon\to 0}[T(x+\epsilon)\bar{T}(x) - \Theta(x+\epsilon)\Theta(x) + \text{derivative terms}]$$

$|\theta_1, \ldots, \theta_n\rangle \equiv |n\rangle$

$P^\mu|n\rangle = P_n^\mu|n\rangle$ $\qquad\qquad$ $\langle m|\Phi(x)|n\rangle = e^{i(P_m^\mu - P_n^\mu)x_\mu}\langle m|\Phi(0)|n\rangle$

$$\langle m|T(x)\bar{T}(0)|n\rangle - \langle m|\Theta(x)\Theta(0)|n\rangle = \langle m|T\bar{T}(0)|n\rangle \qquad \text{if} \quad P_m^\mu = P_n^\mu$$

In particular $\qquad$ $\langle T\bar{T}\rangle = -\langle\Theta\rangle^2$

(originally observed in Fateev, Fradkin, Lukyanov, Zam, Zam, '98; Baseilhac, Stanishkov, '01)

Resonance ambiguity $\quad T\bar{T} \to T\bar{T} + \alpha\,\partial\bar{\partial}\Theta$

All this does not require integrability

$T\bar{T}$ **in massive integrable theories** (GD, Niccoli, '04)

i) asymptotic properties (for $F_n^{T\bar{T}} \neq 0 \quad \forall n$)

$$\lim_{\theta_i \to +\infty} F_n^{T\bar{T}}(\theta_1, \ldots, \theta_n) \sim \exp(2\theta_i) \qquad n > 1$$

$$\lim_{\lambda \to +\infty} e^{-2\lambda} F_n^{T\bar{T}}(\theta_1 + \lambda, \ldots, \theta_k + \lambda, \theta_{k+1}, \ldots, \theta_n) = F_k^{T}(\theta_1, \ldots, \theta_k)\, F_{n-k}^{\bar{T}}(\theta_{k+1}, \ldots, \theta_n)$$

ii) $\langle m|T\bar{T}(0)|n\rangle = \langle m|T(x)\bar{T}(0)|n\rangle - \langle m|\Theta(x)\Theta(0)|n\rangle$ if $P_m^\mu = P_n^\mu$

constrain also asymptotically subleading terms in $F_n^{T\bar{T}}$

iii) $F_n^{T\bar{T}} =$ sum of terms containing $(\sigma_1^{(n)})^i(\sigma_{n-1}^{(n)})^j \qquad i + j \geq 2$

Indeed $\quad T = \partial^2 A \qquad \bar{T} = \bar{\partial}^2 A \qquad \Theta = \partial\bar{\partial} A \qquad$ (A non-local)

$$\Longrightarrow \; T \cdot \bar{T} - \Theta \cdot \Theta = \tfrac{1}{2}\partial^2(A \cdot \bar{T}) + \tfrac{1}{2}\bar{\partial}^2(A \cdot T) - \partial\bar{\partial}(A \cdot \Theta) \qquad A \cdot B \equiv A(x+\epsilon)B(x)$$

Lee-Yang model : i) and ii) uniquely determine $F_n^{T\bar{T}}$, up to the contribution of the resonance $\partial\bar{\partial}\Theta$

i) and ii) overlap consistently $\qquad$ iii) is automatically satisfied

Checked against conformal perturbation theory in V.Belavin, Miroshnichenko,'05

Sinh-Gordon model

$$\mathcal{A}_{shG} = \int d^2x \left[\frac{1}{2}(\partial_\nu\varphi)^2 + \mu\, e^{\sqrt{8\pi}\, b\,\varphi} + \mu'\, e^{-\sqrt{8\pi}\, b\,\varphi}\right]$$

Two interpretation as a perturbed CFT

1) perturbed Gaussian theory: $\Theta \sim \mu \cosh \sqrt{8\pi} b\varphi$ $\qquad \varphi \to -\varphi$ invariance

2) perturbed Liouville theory: $\Theta \sim \mu'\, e^{-\sqrt{8\pi}\, b\,\varphi}$

Integrable theory with a single particle and no bound states

$$S(\theta) = \frac{\tanh\frac{1}{2}(\theta + i\xi)}{\tanh\frac{1}{2}(\theta - i\xi)} \qquad \xi = -\frac{\pi b^2}{1+b^2}$$

Connection with massive minimal models

For $\xi = 2\pi/(2N+1)$, $S(\theta)$ coincides with the amplitude for the lightest particle of the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2,2N+3}$, $N = 1, 2, \ldots$ (Lee-Yang for $N = 1$)

Bound states appear for these positive values of ξ

Form factors of primary operators

(Fring, Mussardo, Simonetti, '92; Koubek, Mussardo, '93)

$$F_n^{\Phi}(\theta_1,\ldots,\theta_n) = U_n^{\Phi}(\theta_1,\ldots,\theta_n) \prod_{i<j} \frac{\mathcal{F}(\theta_i-\theta_j)}{\cosh\frac{\theta_i-\theta_j}{2}}$$

$$\mathcal{F}(\theta) = \mathcal{N}(\xi)\,\exp\left[2\int_0^\infty \frac{dt}{t}\, q_\xi(t)\,\frac{\sinh\frac{t}{2}}{\sinh^2 t}\,\sin^2\frac{(i\pi-\theta)t}{2\pi}\right]$$

$$q_\xi(t) = -4\sinh\frac{\xi t}{2\pi}\,\sinh\left[\left(1+\frac{\xi}{\pi}\right)\frac{t}{2}\right] \qquad \mathcal{N}(\xi) = \exp\left[-\int_0^\infty \frac{dt}{t}\, q_\xi(t)\,\frac{\sinh\frac{t}{2}}{\sinh^2 t}\right]$$

The primary operators $V_a = \exp(-\sqrt{8\pi}ab\,\varphi)$ correspond to

$$U_n^{V_a}(\theta_1,..,\theta_n) = \langle V_a\rangle \left(\frac{-8\sin\xi}{2^n\mathcal{N}(\xi)}\right)^{n/2}\left(\frac{1}{\sigma_n^{(n)}}\right)^{(n-1)/2}[a]\,\det M^{(n)}(a)$$

$$[m] \equiv \frac{\sin(m\xi)}{\sin\xi} \qquad M_{i,j}^{(n)}(a) = [a+i-j]\,\sigma_{2i-j}^{(n)} \qquad i,j = 1,\ldots,n-1$$

For $\xi = 2\pi/(2N+1)$ the bound state equations select $a = 0,1,\ldots,N$, in correspondence with the $N+1$ primaries of the perturbed $\mathcal{M}_{2,2N+3}$ models

Form factors of $T\bar{T}$ (GD, Niccoli, '06)

We impose the constraints i)–iii) above, and also require that for $\xi = 2\pi/(2N+1)$ the result is compatible with the bound state structure of the perturbed $\mathcal{M}_{2,2N+3}$ models. We obtain

$$F_n^{T\bar{T}} = a\, F_n^{\partial^2\bar{\partial}^2\Theta} + F_n^{\mathcal{K}} + c\, F_n^{\partial\bar{\partial}\Theta} + d\, F_n^{\Theta} + e\, F_n^{I}$$

$$a = \frac{\langle\Theta\rangle}{m^4} \qquad d = -2\,\langle\Theta\rangle \qquad e = \langle\Theta\rangle^2 \qquad \left(\langle\Theta\rangle = -\frac{\pi m^2}{8\sin\xi}\right)$$

$F_n^{\mathcal{K}}$ is a kernel solution which vanishes for $n = 0, 1, 2$

$$U_n^{\mathcal{K}}(\theta_1, \ldots, \theta_n) = -\langle\Theta\rangle^2 \left(\frac{-8\sin\xi}{2^n \mathcal{N}(\xi)}\right)^{n/2} \left(\frac{1}{\sigma_n^{(n)}}\right)^{(n-1)/2} \tilde{Q}_n^{\mathcal{K}}(\theta_1, \ldots, \theta_n)$$

$$\tilde{Q}_3^{\mathcal{K}} = \tfrac{1}{\sigma_3^2}(\sigma_1^2\sigma_2^2 - \sigma_2^3 - \sigma_1^3\sigma_3 - (1 + 2\cos[2\xi])\,\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2 - \sigma_3)$$

$$\tilde{Q}_4^{\mathcal{K}} = \frac{1}{\sigma_4^2}(\sigma_1^2\sigma_3^2 - \sigma_2^2\sigma_4 - 2\cos[\xi](1 + 4\cos[\xi] + 2\cos[2\xi])\,\sigma_1\sigma_3\sigma_4 + 4\cos[\xi]^2\sigma_4^2) \times (\sigma_1\sigma_3\sigma_2 - \sigma_3^2 - \sigma_1^2\sigma_4)$$

$$
\tilde{Q}_5^{\mathcal{K}} = \frac{1}{\sigma_5^2}(-\sigma_1^4\sigma_3\sigma_4^2\sigma_5 + \sigma_4^2\sigma_5(\sigma_2^2\sigma_3 - 4\cos[\xi]^2\sigma_3\sigma_4 - \sigma_2\sigma_5) + \sigma_1\sigma_4\sigma_5(-\sigma_2^3\sigma_4 +
$$

$$
\sigma_2(2(2+\cos[2\xi])\sigma_4^2 - (7 + 8\cos[\xi] + 8\cos[2\xi] + 4\cos[3\xi] + 2\cos[4\xi])\sigma_3\sigma_5) +
$$

$$
4\cos[\xi]((1 + 4\cos[\xi] + 2\cos[2\xi] + \cos[3\xi])\sigma_3^2\sigma_4 + (1 + 2\cos[\xi] + 2\cos[2\xi])\sigma_5^2))
$$

$$
+\sigma_1^2(2(4 + 4\cos[\xi] + 5\cos[2\xi] + 2\cos[3\xi] + \cos[4\xi])\sigma_2^2\sigma_4\sigma_5^2 - \sigma_5(\sigma_3^3\sigma_4 + 8\cos[\frac{\xi}{2}]^2 (
$$

$$
\cos[\xi] + 2\cos[2\xi] + \cos[3\xi])\sigma_4^2\sigma_5 + \sigma_3\sigma_5^2) - \sigma_2(\sigma_4^4 + 2(5 + 4\cos[\xi] + 6\cos[2\xi]
$$

$$
+2\cos[3\xi] + \cos[4\xi])\sigma_3\sigma_4^2\sigma_5 - \sigma_3^2\sigma_5^2)) + \sigma_1^3(2\sigma_4\sigma_5((3 + 4\cos[\xi] + 4\cos[2\xi] +
$$

$$
2\cos[3\xi] + \cos[4\xi])\sigma_4^2 + (2 + \cos[2\xi])\sigma_3\sigma_5) + \sigma_2(\sigma_3\sigma_4^3 - 4\cos[\xi]^2\sigma_5^3)))
$$

Determined explicitely up to $n = 9$

– The perturbed Gaussian case is obtained setting $F_n^{T\bar{T}} = 0$ for n odd

– For $\xi = 2\pi/(2N+1)$, $T\bar{T}$ for the perturbed $\mathcal{M}_{2,2N+3}$ models is obtained

Conclusions

– The program of determining the full operator space from the particle dynamics is not far from being accomplished in integrable quantum field theory

– The analyticity requirements (form factor equations) select a space of solutions with the correct dimensionality

– They need to be supplemented by asymptotic conditions which play a crucial role in the identification of operators

– At present we can classify the solutions according to the level and identify the primaries and all descendants up to a certain level

– What is missing is the identification at arbitrarily high level for those operators which cannot be obtained exploiting the conserved quantities