The antiferromagnetic Potts model on the square lattice

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Introduction

Overview

1. The antiferromagnetic Potts model as an integrable six-vertex model

2. The $\mathcal{R}$-matrix acting on double-edges

3. Coordinate Bethe ansatz

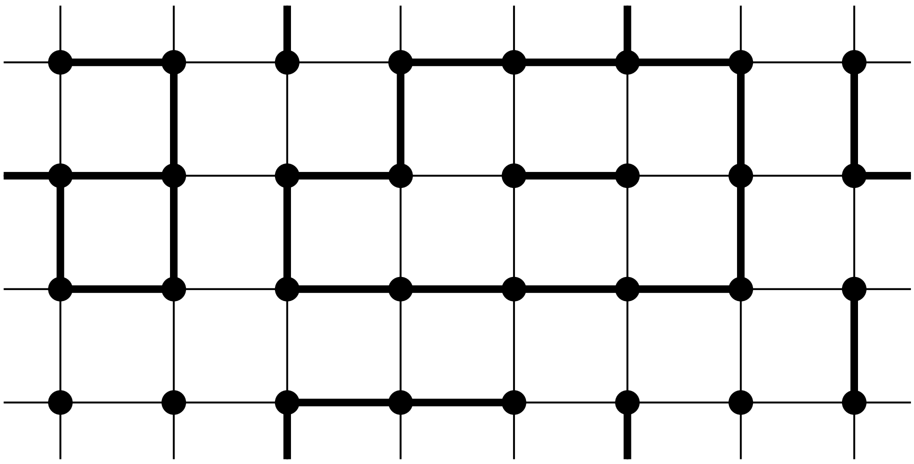
4. Low-energy spectrum

5. Relation to σ-models

6. Geometrical formulation

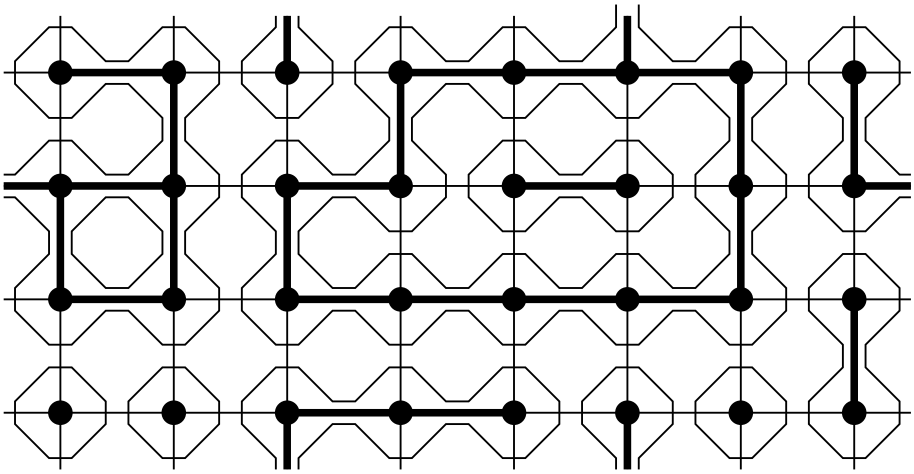
1. The antiferromagnetic Potts model as an integrable six-vertex model

Fortuin-Kasteleyn cluster model



$$Z_{\mathsf{FK}} = \sum_{\text{clusters}} Q^{\#\text{clusters}} \, v_1^{\#\text{vert. bonds}} \, v_2^{\#\text{hor. bonds}}$$

Equivalent loop model



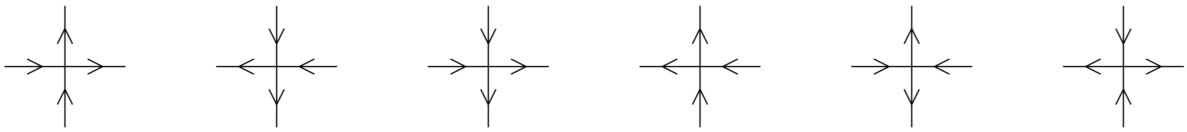
$$Z_{\text{loops}} = \sum_{\text{loops}} \sqrt{Q}^{\#\text{loops}} \; x_1^{\#\text{vert. bonds}} \; x_2^{\#\text{hor. bonds}}$$

where :

$$x_r = \frac{v_r}{\sqrt{Q}}$$

$$\sqrt{Q} = 2\cos\gamma \qquad 0 \le \gamma \le \frac{\pi}{2}$$

Equivalent six-vertex model



ω_1 ω_2 ω_3 ω_4 ω_5 ω_6

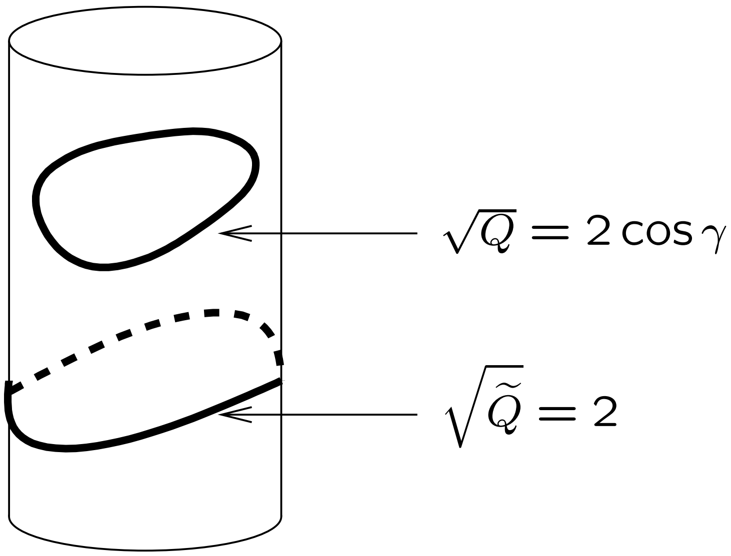
Vertices on the even sublattice :

$$\omega_1, \ldots, \omega_6 = 1, 1, x_1, x_1, e^{i\gamma/2} + x_1 e^{-i\gamma/2}, e^{-i\gamma/2} + x_1 e^{i\gamma/2}$$

Vertices on the odd sublattice :

$$\omega_1', \ldots, \omega_6' = x_2, x_2, 1, 1, e^{-i\gamma/2} + x_2 e^{i\gamma/2}, e^{i\gamma/2} + x_2 e^{-i\gamma/2}$$

Loop weights in the *untwisted* six-vertex model on a cylinder

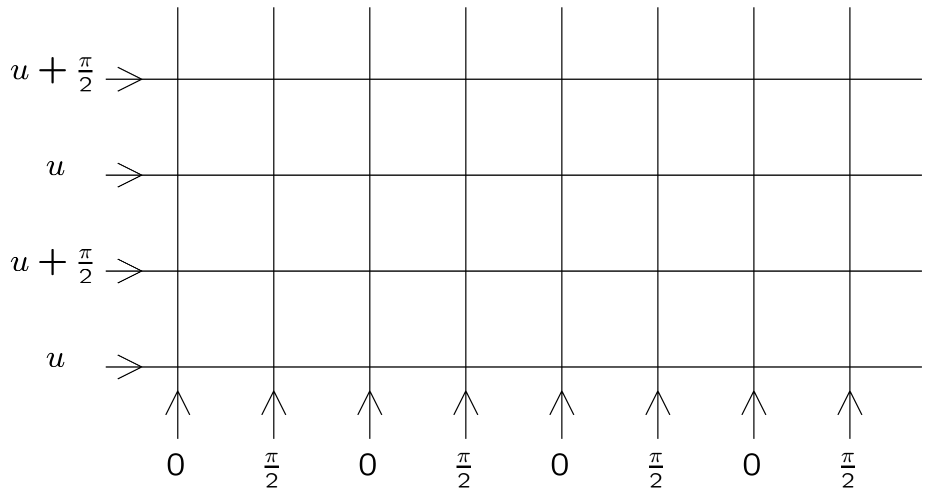


Position of the antiferromagnetic critical point

$$x_1 = \frac{\sin u}{\sin(\gamma - u)} \;, \qquad x_2 = -\frac{\cos(\gamma - u)}{\cos u}$$

$$\left.\begin{array}{rrcl} & \gamma < & u & < \frac{\pi}{2} \\ or & \gamma - \frac{\pi}{2} < & u & < 0 \end{array}\right\} \Rightarrow \; x_1 < 0 \;,\; x_2 < 0$$

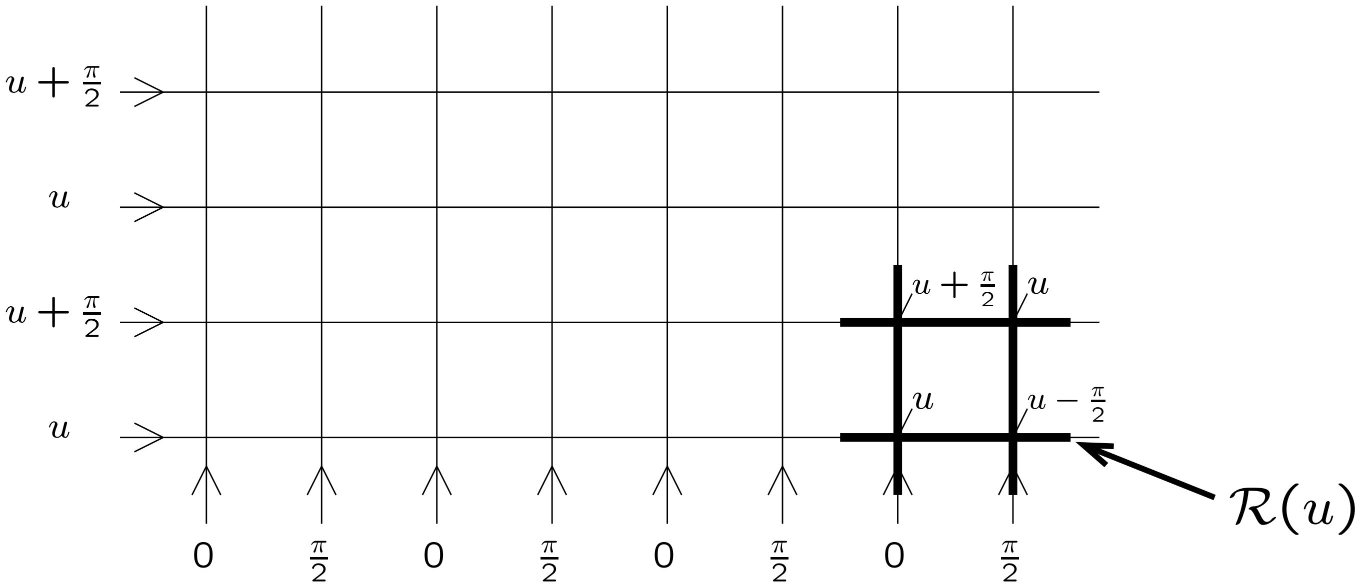
Spectral parameter structure :



(Baxter, 1982)

2. The $\mathcal{R}$-matrix acting on double-edges

The $\mathcal{R}$-matrix : definition, symmetries



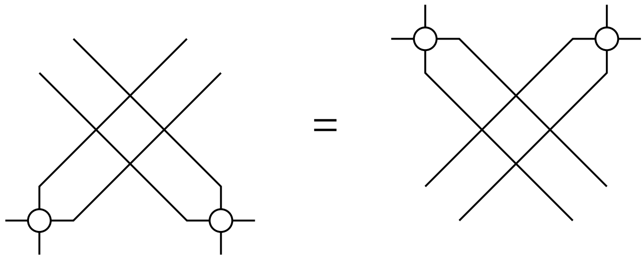
Total magnetization conservation :

$$[\mathcal{R}(u), S^z] = 0$$

"Charge" conservation :

$$c \equiv (-\cos\gamma)^{-1} R_{6V}(\pi/2) \ , \qquad c^2 = 1$$

$$[\mathcal{R}(u), c \otimes c] = 0$$

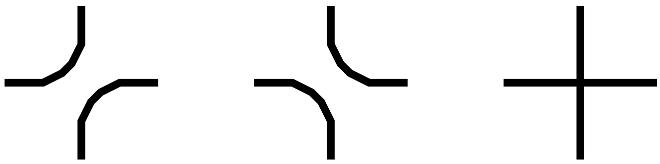


The $\mathcal{R}$-matrix : limiting cases

a. The limit $\gamma \to \pi/2$

$$\begin{cases} \gamma = \pi/2 - \epsilon & \epsilon \to 0^+ \\ u = \pi/2 - \epsilon w & 0 \leq w \leq 1 \end{cases}$$

$$\begin{cases} \widetilde{\mathcal{R}}_{\gamma\to\pi/2} = (1-w)1 + wE + w(1-w)P \\ \\ \text{loop fugacity } q = 0 \end{cases}$$



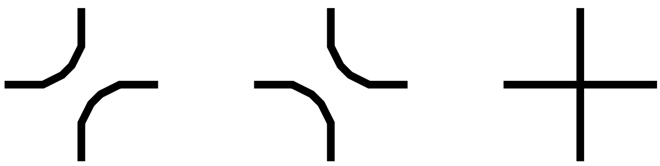
1 $\quad E \quad P$

Integrable $OSP(2|2)$ loop model
(Martins, Nienhuis, Rietman, 1998)

b. The limit $\gamma \to 0$

$$\begin{cases} \gamma = \epsilon & \epsilon \to 0^+ \\ u = -\epsilon w & w \geq 0 \end{cases}$$

$$\begin{cases} \widetilde{\mathcal{R}}_{\gamma \to 0} = (1+w)1 - wE + w(1+w)P \\ \text{loop fugacity } q = 4 \end{cases}$$



1 E P

Integrable $SO(4)$ loop model
(Martins, Nienhuis, Rietman, 1998)

Hamiltonian limit of the transfer matrix ($u \to 0^-$)

$$\mathcal{R}(u) = -\frac{1}{4}\sin^2 2\gamma \; c \otimes c + u \; \delta\mathcal{R} + O(u^2)$$

$$\downarrow$$

$$\mathcal{T} = \left(-\frac{1}{4}\sin^2 2\gamma\right)^N e^{-2iP} \left[1 + \left(\frac{1}{2}\sin 2\gamma\right)^{-1} u \; \mathcal{H} + O(u^2)\right]$$

$$\mathcal{H} = \sum_{m=1}^{2N} \left[-\cos 2\gamma \; 1 + 2\cos\gamma \; E_m - (E_{m+1}E_m + E_m E_{m+1}) \right]$$

E_m : Temperley-Lieb generators acting on single-edges

3. Coordinate Bethe ansatz

Coordinate Bethe ansatz in additive parametrization (1)

a. Bethe equations :

$$\exp\left[2iNk(\alpha_j)\right] = -\prod_{l=1}^{r} \exp\left[-i\phi(\alpha_j, \alpha_l)\right] \qquad (\mathrm{BE}_j)$$

where :

$$\begin{aligned} \exp\left[2ik(\alpha)\right] &= \frac{\sinh(\alpha + i\gamma)}{\sinh(\alpha - i\gamma)} \\ \exp\left[i\phi(\alpha, \alpha')\right] &= \frac{\sinh\frac{1}{2}(\alpha - \alpha' - 2i\gamma)}{\sinh\frac{1}{2}(\alpha - \alpha' + 2i\gamma)} \end{aligned}$$

b. Total energy :

$$\mathcal{E}(\alpha_1, \ldots, \alpha_r) = -2N\cos 2\gamma + \sum_{j=1}^{r} \frac{2\sin^2 2\gamma}{\cosh 2\alpha_j - \cos 2\gamma}$$

(Baxter, 1971)

Coordinate Bethe ansatz in additive parametrization (2)

c. Bethe states :

One-particle states

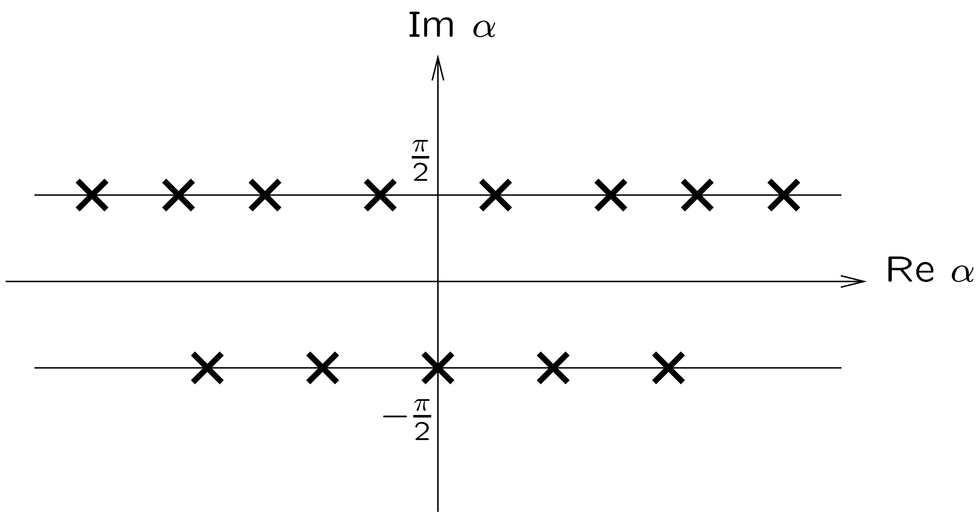
$$\begin{cases} \varphi_\alpha(2m-1) = \quad \cosh\frac{1}{2}(\alpha - i\gamma)\ e^{2ik(\alpha)m} \\ \varphi_\alpha(2m) = -\sinh\frac{1}{2}(\alpha + i\gamma)\ e^{2ik(\alpha)m} \end{cases}$$

r-particle states

$$\varphi_{\alpha_1,\ldots,\alpha_r}(x_1,\ldots,x_r)$$

(Baxter, 1971)

Two types of particles



$$e^{2iNk(\lambda_j)} = (-1)^{r_+-1} \prod_{l=1}^{r_+} e^{-i\Theta_1(\lambda_j-\lambda_l)} \prod_{l=1}^{r_-} e^{-i\Theta_{-1}(\lambda_j-\mu_l)}$$

$$e^{2iNk(\mu_j)} = (-1)^{r_--1} \prod_{l=1}^{r_+} e^{-i\Theta_{-1}(\mu_j-\lambda_l)} \prod_{l=1}^{r_-} e^{-i\Theta_1(\mu_j-\mu_l)}$$

$$\exp[2ik(\lambda)] = \frac{\cosh(\lambda+i\gamma)}{\cosh(\lambda-i\gamma)}$$

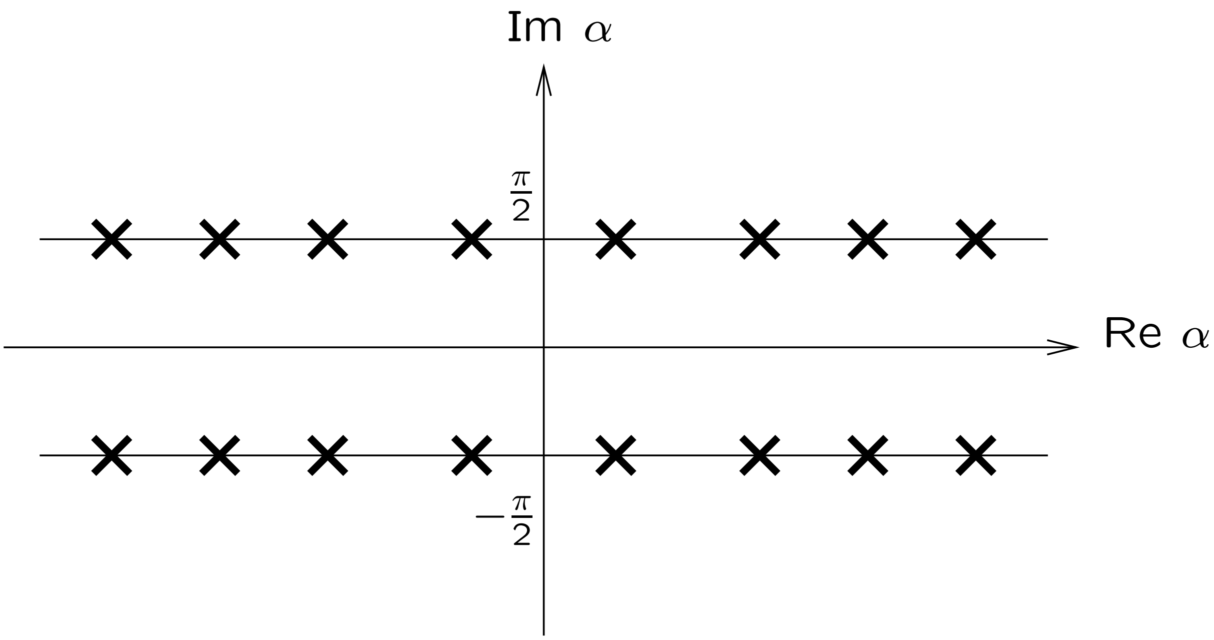
$$\exp[i\Theta_1(\lambda)] = -\frac{\sinh\frac{1}{2}(\lambda-2i\gamma)}{\sinh\frac{1}{2}(\lambda+2i\gamma)}$$

$$\exp[i\Theta_{-1}(\lambda)] = \frac{\cosh\frac{1}{2}(\lambda-2i\gamma)}{\cosh\frac{1}{2}(\lambda+2i\gamma)}$$

$$\mathcal{E}(\lambda_1,\ldots,\lambda_{r_+}|\mu_1,\ldots,\mu_{r_-}) = -2N\cos 2\gamma - \sum_{j=1}^{r_+} \frac{2\sin^2 2\gamma}{\cosh 2\lambda_j + \cos 2\gamma} - \sum_{j=1}^{r_-} \frac{2\sin^2 2\gamma}{\cosh 2\mu_j + \cos 2\gamma}$$

(Jacobsen, Saleur, 2006)

XXZ subspectrum



Solutions with $r_+ = r_-$, $\lambda_j = \mu_j$

$$(\mathrm{BE}_j) \Leftrightarrow \exp\left[iNk_0(\lambda_j)\right] = -\prod_{l=1}^{r} \exp\left[-i\phi_0(\lambda_j,\lambda_l)\right]$$

$$\mathcal{E}(\lambda_1,\ldots,\lambda_r|\lambda_1,\ldots,\lambda_r) = 2\mathcal{E}_0(\lambda_1,\ldots,\lambda_r)$$

$\mathcal{E}_0$ eigenvalue of :

$$H_0 = -1/2\sum_{m=1}^{N}\left[\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y - \cos\gamma_0\ (\sigma_m^z\sigma_{m+1}^z + 1)\right]$$

$$\gamma_0 = \pi - 2\gamma$$

$\mathbb{Z}/2\mathbb{Z}$ symmetry

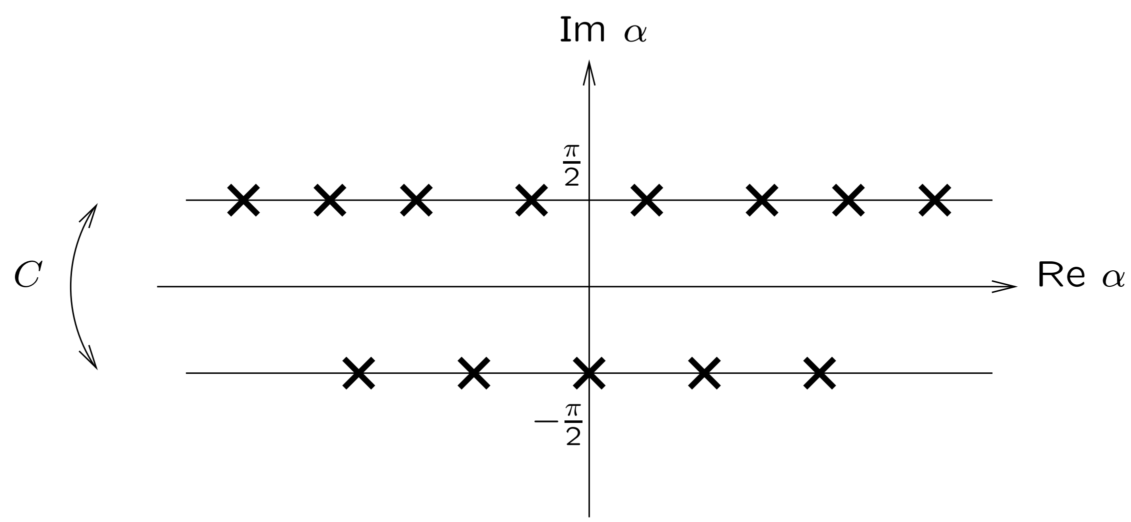
Charge operator :

$$C = \prod_{j=1}^{N} c_{2j-1} \ , \qquad C^2 = 1$$

$$[\mathcal{H}, C] = 0$$

Action on the Bethe states :

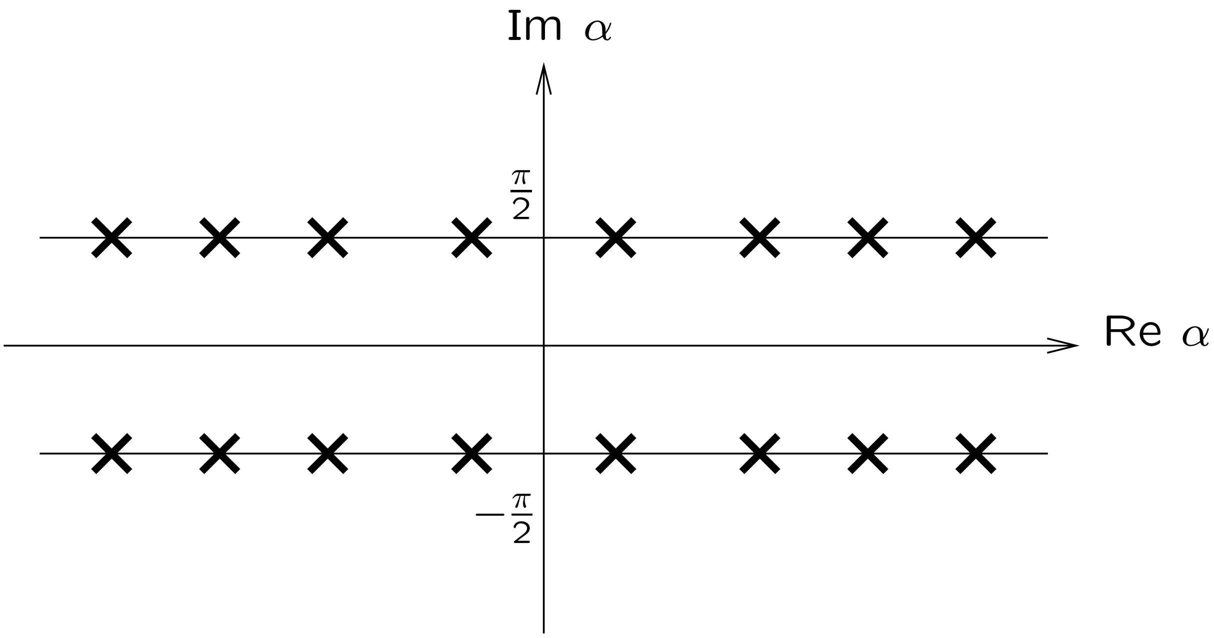
$$C\varphi_{(\lambda_1,\ldots,\lambda_{r_+}|\mu_1,\ldots,\mu_{r_-})} = e^{i\frac{\pi}{2}(r_+ - r_-)} \ \varphi_{(\mu_1,\ldots,\mu_{r_-}|\lambda_1,\ldots,\lambda_{r_+})}$$



4. Low-energy spectrum

Ground state

$$r_+ = r_- = N/2$$



Fermi velocity of particle-hole excitations :

$$\mathcal{E}(k) \simeq v|k| \ , \qquad v = 2v_0 = \frac{2\pi \sin 2\gamma}{\pi - 2\gamma}$$

Central charge :

$$c = 2$$

Excited states

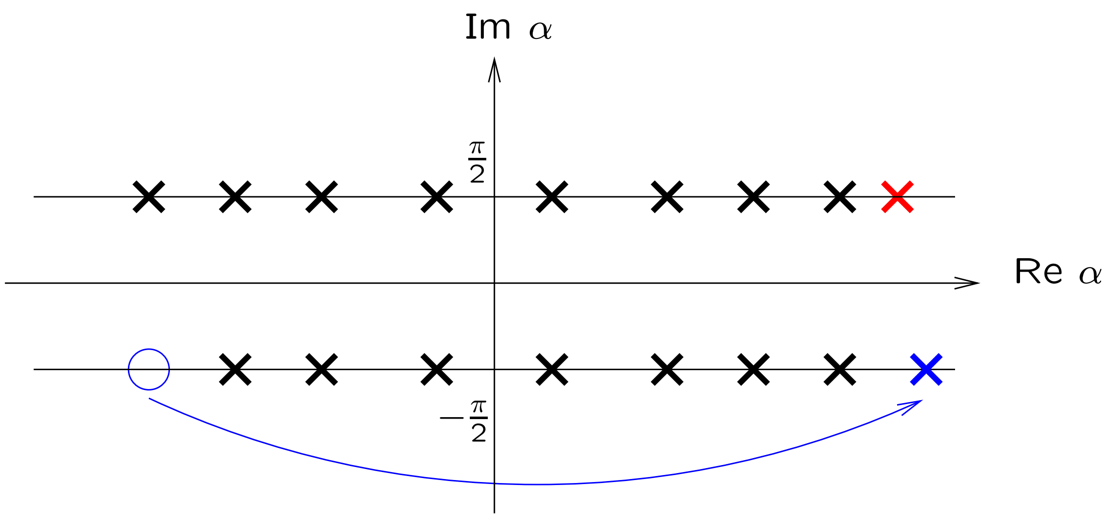
Particle excitations :

$$\begin{cases} r_+ &= N/2 - n_+ \\ r_- &= N/2 - n_- \end{cases}$$

Backscattering excitations :

Shifting of the Bethe integers I_j^+ by m_+ units

Shifting of the Bethe integers I_j^- by m_- units



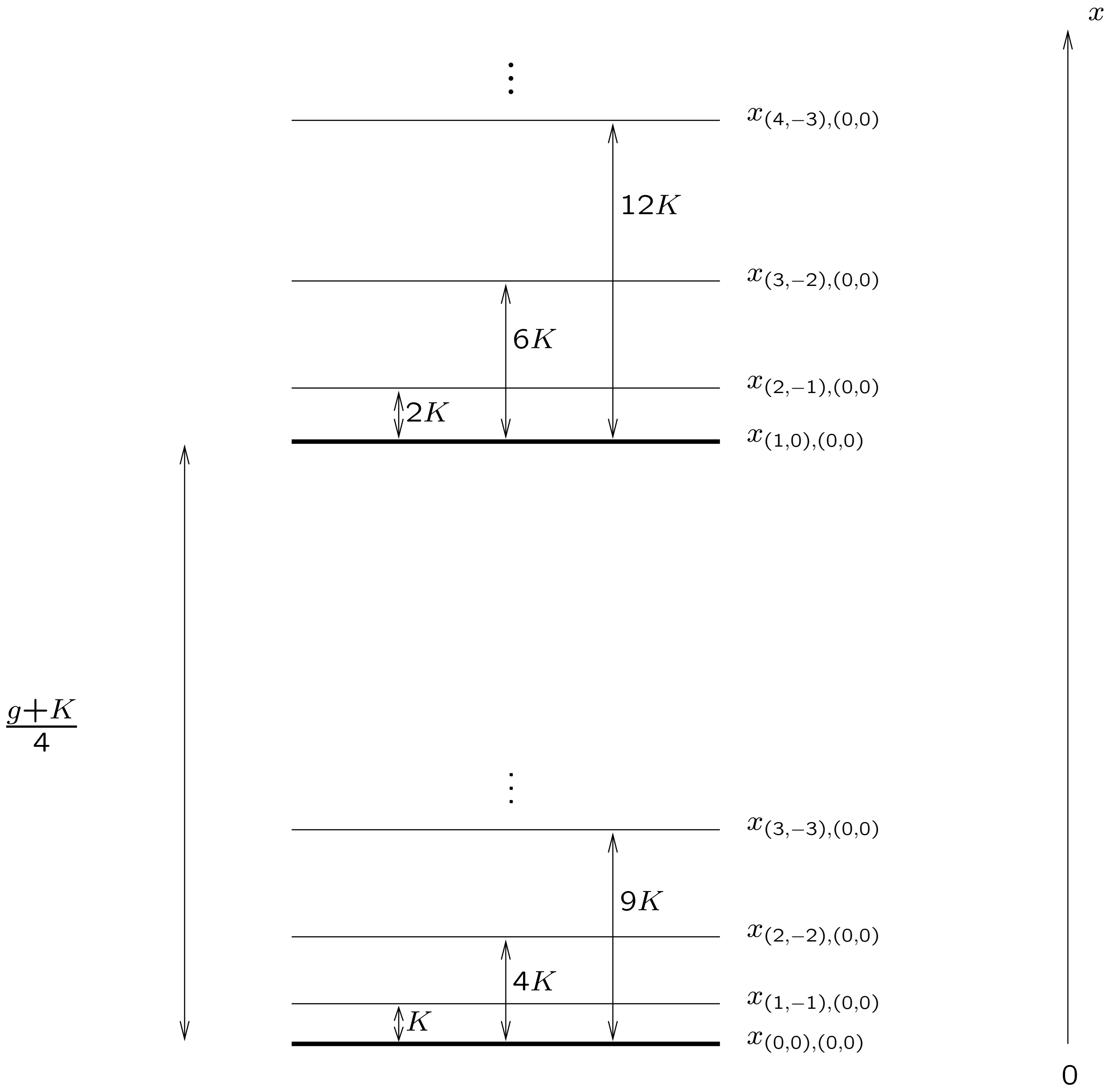
Physical exponents :

$$\begin{aligned} x_{(n_+,n_-),(m_+,m_-)} &= \frac{1}{4} g (n_+ + n_-)^2 + \frac{1}{4g}(m_+ + m_-)^2 \\ &+ \frac{1}{4} K(N)(n_+ - n_-)^2 + \frac{1}{4K(N)}(m_+ - m_-)^2 \end{aligned}$$

$$g = 2\gamma/\pi \ , \qquad K(N) \underset{N\to\infty}{\longrightarrow} 0$$

Structure of the low-energy spectrum

$$\mathcal{E} = \mathcal{E}_0 + \frac{2\pi v x}{2N} + o(1/N)$$

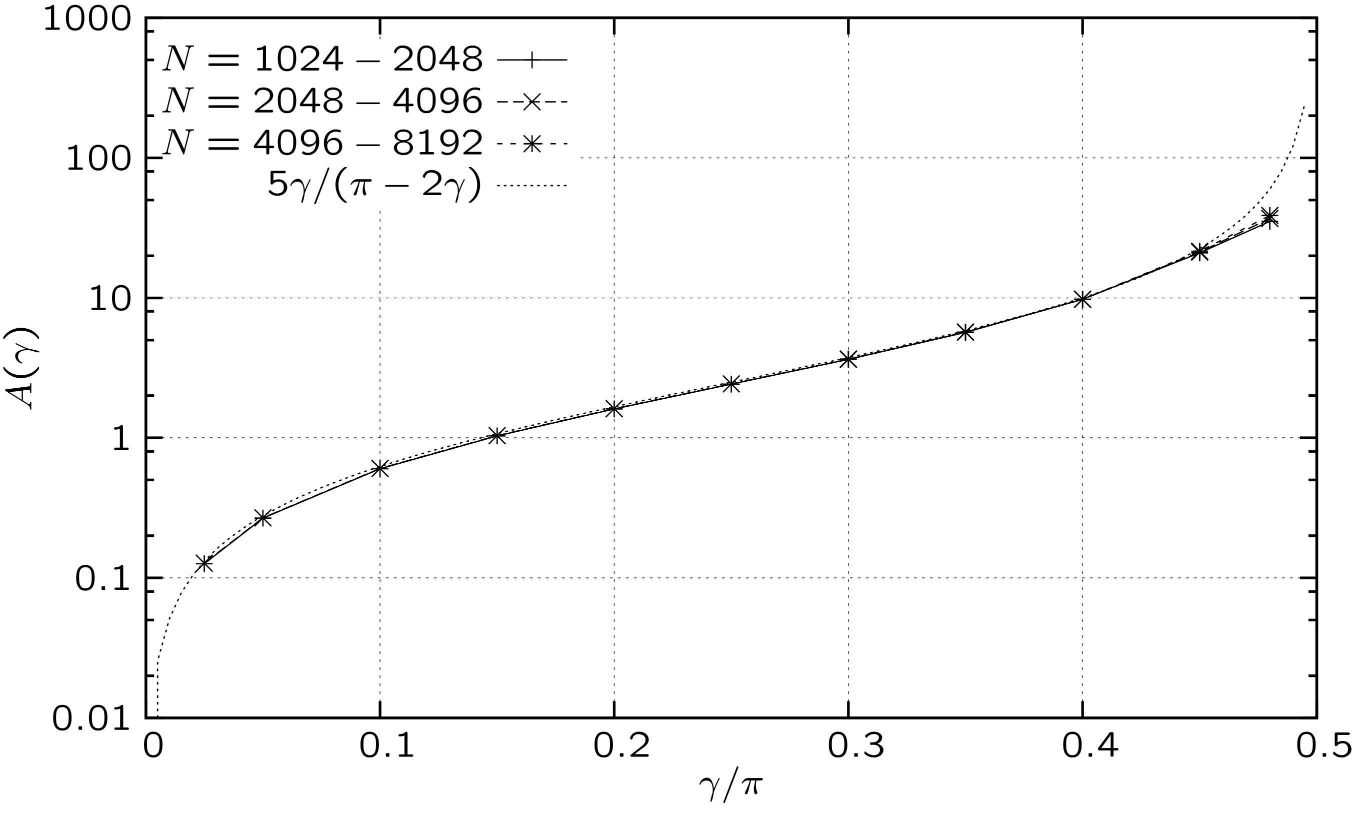


Scaling law of the coupling constant $K(N)$

$$K(N) \simeq \frac{A(\gamma)}{[B(\gamma) + \log N]^2} \qquad (\gamma < \pi/2)$$

$$K(N) \simeq \frac{A'}{B' + \log N} \qquad (\gamma = \pi/2)$$

Scaling function $A(\gamma)$



5. Relation to σ-models

$OSP(2|2)$ σ-model

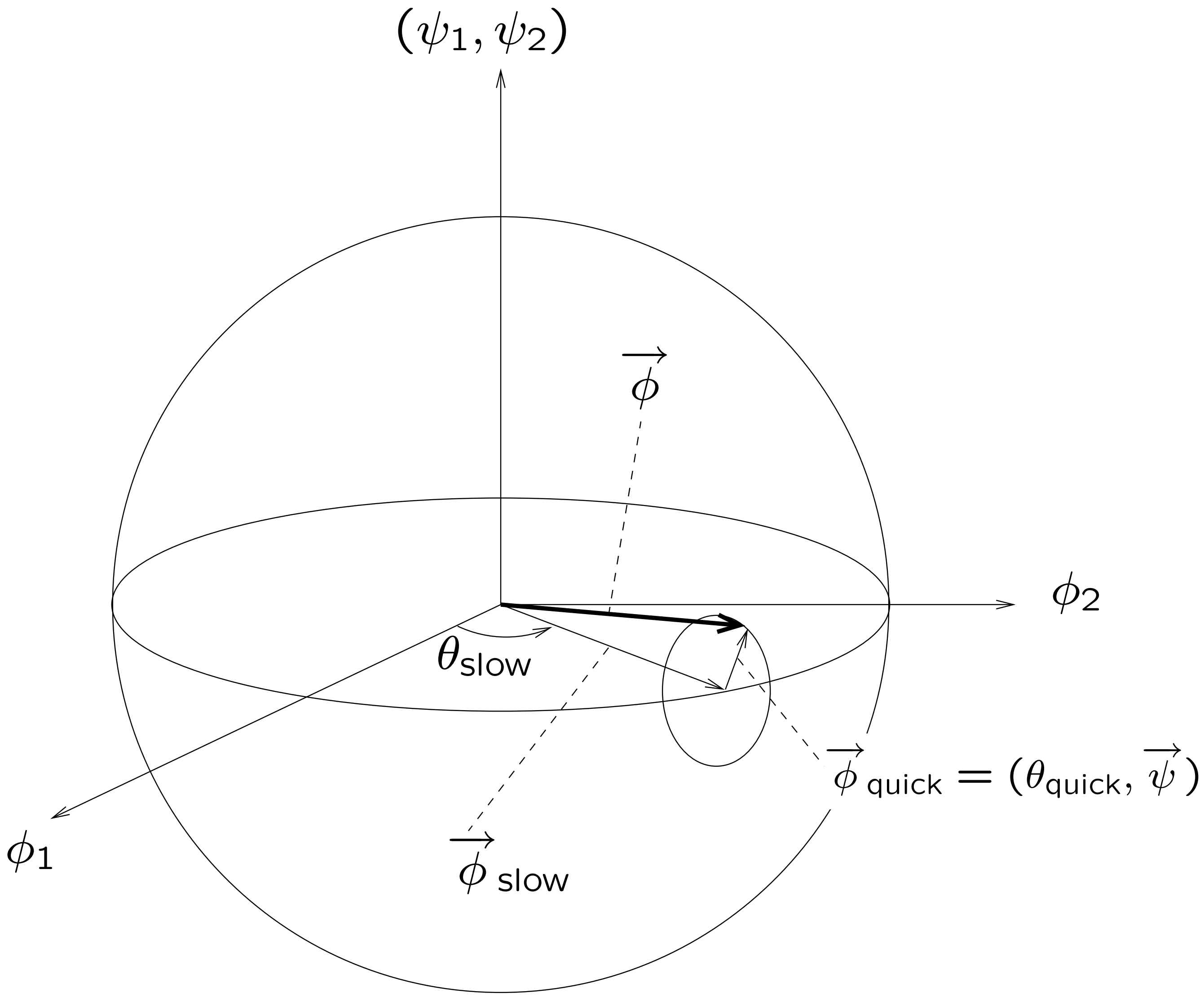
a. Definition

Four-dimensional field :

$$\begin{aligned}\overrightarrow{\phi} &= (\phi_1, \phi_2, \psi_1, \psi_2) \\ \overrightarrow{\phi} \cdot \overrightarrow{\phi}' &\equiv \phi_1\phi_1' + \phi_2\phi_2' + \psi_1\psi_2' - \psi_2\psi_1'\end{aligned}$$

Action defined on the "supersphere" :

$$S = \frac{1}{2G}\int d^2x\ \partial_\mu \overrightarrow{\phi} \cdot \partial_\mu \overrightarrow{\phi}\ , \qquad \overrightarrow{\phi} \cdot \overrightarrow{\phi} = 1$$



$OSP(2|2)$ σ-model

b. Long-distance behaviour

RG equation :

$$\frac{dG}{d\log s} = -2 \times \frac{G^2}{2\pi} \quad \Rightarrow \quad G \simeq \frac{\pi}{\log s}$$

The long-distance behaviour is described by the limit $G \to 0$

Rescaling :

$$|\nabla\phi| \sim \sqrt{G}$$

$$\psi_1 \to \sqrt{G/(4\pi)}\ \psi_1\ ,\ \psi_2 \to \sqrt{G/(4\pi)}\ \psi_2\ ,\ \theta \to \sqrt{G/(4\pi)}\ \theta$$

$$\begin{aligned} S \;=\; & \frac{1}{8\pi}\int d^2x\Big[(\partial_\mu\theta)^2\left(1-\frac{G}{2\pi}\psi_1\psi_2\right) \\ & +2\partial_\mu\psi_1\partial_\mu\psi_2+\frac{G}{\pi}\psi_1\psi_2\partial_\mu\psi_1\partial_\mu\psi_2\Big] \end{aligned}$$

Asymptotic free theory :

- ψ_1, ψ_2 free symplectic fermions, combining to a free compact boson X of radius : $R_1 = 1$

- θ free compact boson of radius $R_2 = \sqrt{4\pi/G}$

$OSP(2|2)$ σ-model

c. Excitation spectrum

Scaling law for the compactification radius :

$$(R_2)^2 \;=\; 4\pi/G \;\simeq\; 4\log s \;\rightarrow\; \infty$$

Excitation spectrum :

$$x_{(e_1,m_1),(e_2,m_2)} = (e_1)^2 + \frac{(m_1)^2}{4} \quad + \quad \frac{(e_2)^2}{(R_2)^2} + \frac{(m_2)^2(R_2)^2}{4}$$

$$e_1, m_1, e_2, m_2 \in \mathbb{Z}$$

Spectrum of $\mathcal{H}$ for $\gamma \to \pi/2$:

$$\begin{aligned} x_{(n_+,n_-),(m_+,m_-)} \quad &= \quad \frac{1}{4}(n_+ + n_-)^2 + \frac{1}{4}(m_+ + m_-)^2 \\ &+ \quad \frac{1}{4}K(N)(n_+ - n_-)^2 + \frac{1}{4K(N)}(m_+ - m_-)^2 \end{aligned}$$

$$K(N) \simeq 1/\log N$$

Identification in the case $m_+ = m_- = m$:

$$\left\{ \begin{array}{rcl} e_1 &=& \frac{1}{2}(m_+ + m_-) = m \\ m_1 &=& n_+ + n_- \end{array} \right. \qquad \left\{ \begin{array}{rcl} e_2 &=& n_+ - n_- \\ m_2 &=& \frac{1}{2}(m_+ - m_-) = 0 \end{array} \right.$$

Effective field theory in the case $\gamma < \pi/2$

- X free compact boson of finite radius :

$$(R_1)^2 = \frac{2\gamma}{\pi}$$

- θ free compact boson of radius :

$$(R_2)^2 \simeq \frac{4(\log s)^2}{A(\gamma)}$$

Numerical guess :

$$A(\gamma) = \frac{a\gamma}{\pi - 2\gamma}\ , \qquad a \simeq 5$$

6. Geometrical formulation

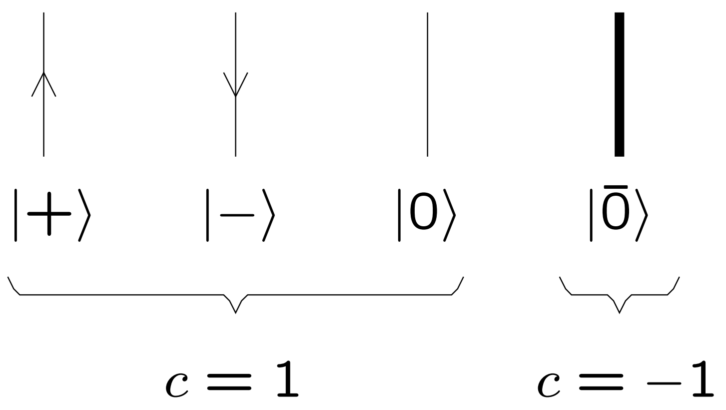
States living on double-edges

Basis change :

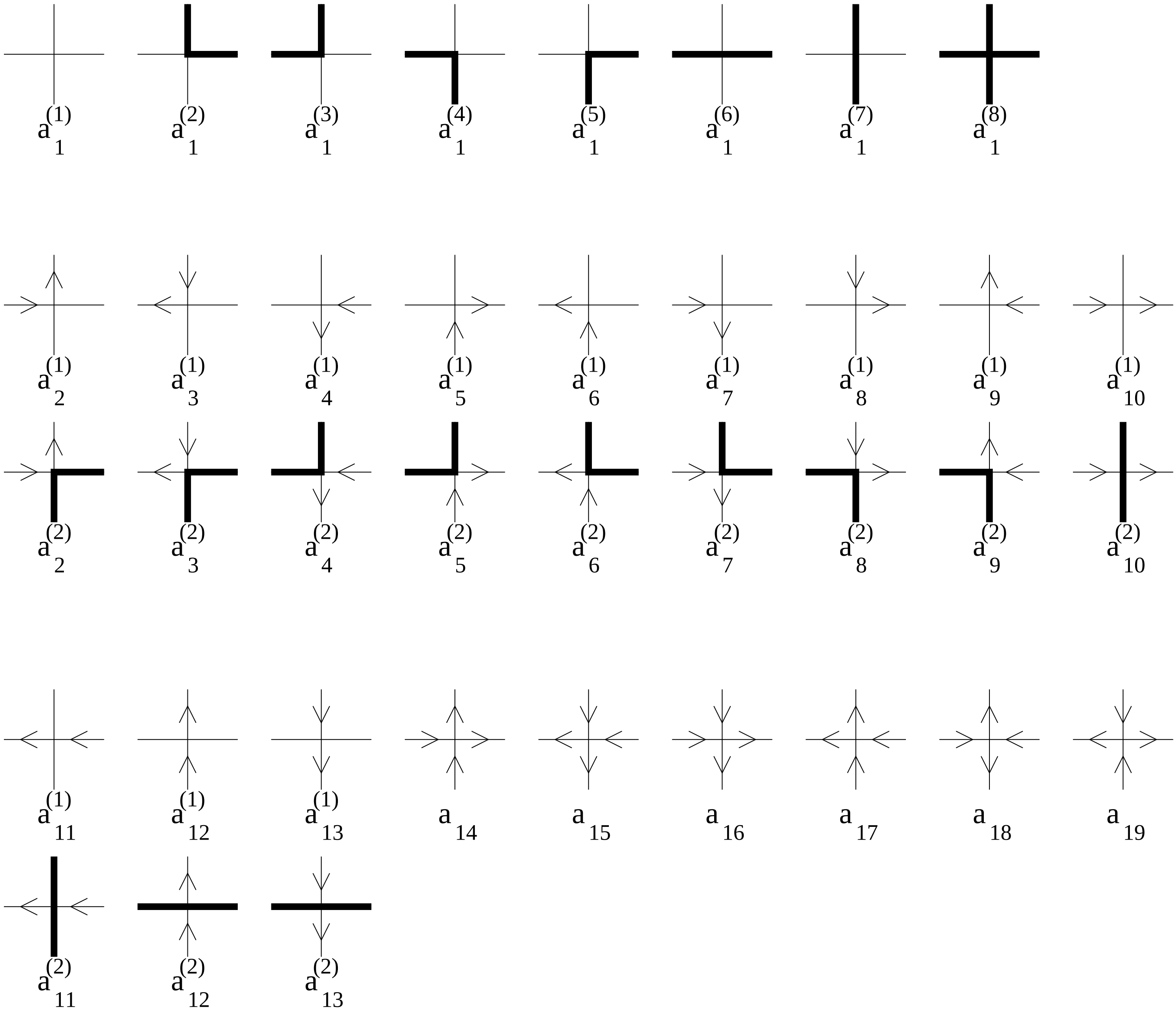
$$\Big\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\Big\} \to \Big\{|+\rangle, |0\rangle, |\bar{0}\rangle, |-\rangle\Big\}$$

$$\begin{aligned}
|+\rangle &= |\uparrow\uparrow\rangle \\
|-\rangle &= |\downarrow\downarrow\rangle \\
|0\rangle &= \frac{1}{\sqrt{2\cos\gamma}}\left(e^{i\gamma/2}|\uparrow\downarrow\rangle - e^{-i\gamma/2}|\downarrow\uparrow\rangle\right) \\
|\bar{0}\rangle &= \frac{1}{\sqrt{2\cos\gamma}}\left(e^{-i\gamma/2}|\uparrow\downarrow\rangle + e^{i\gamma/2}|\downarrow\uparrow\rangle\right)
\end{aligned}$$

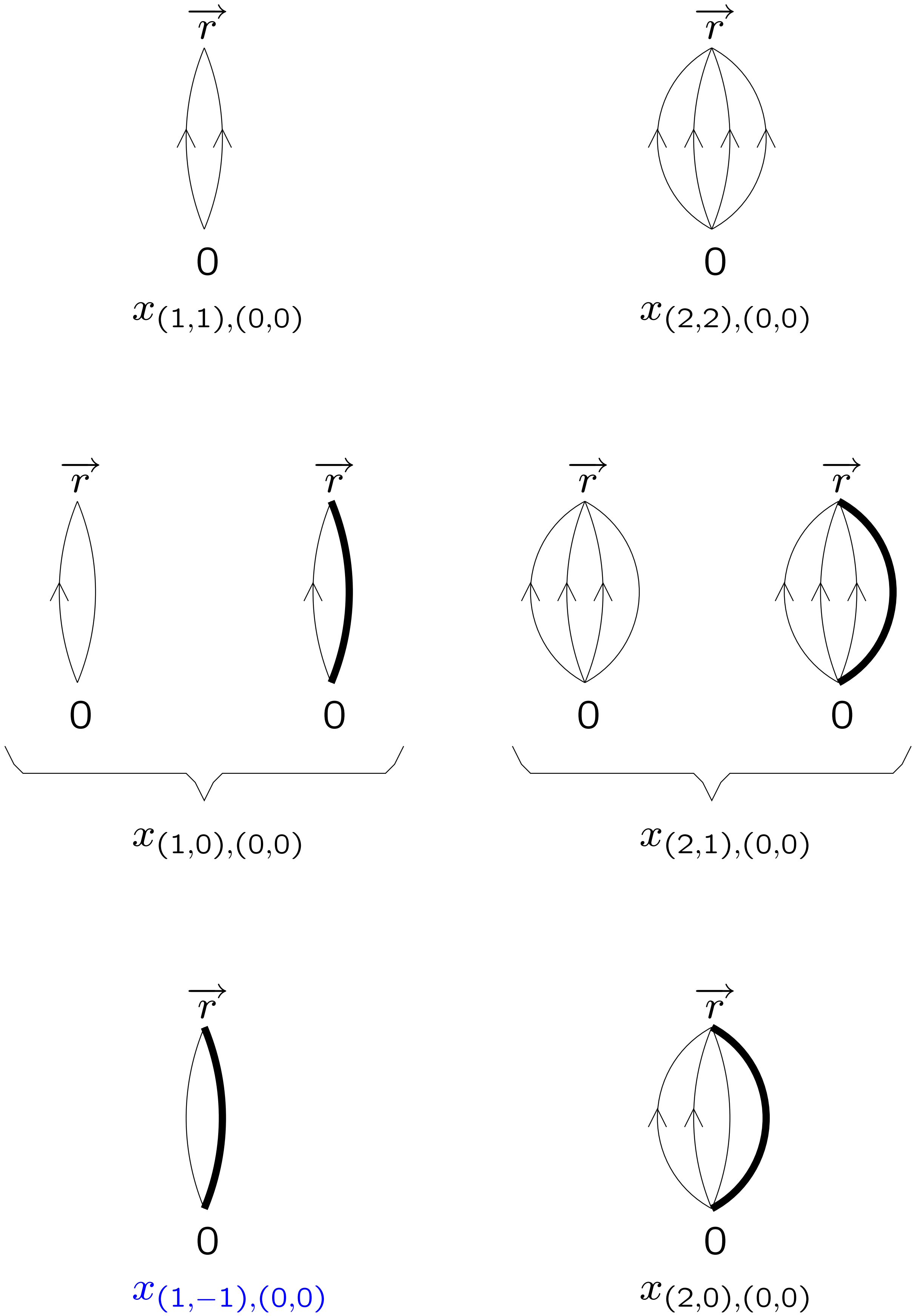
Graphical representation :



The $\mathcal{R}$-matrix as a 38-vertex model



Watermelon correlation functions



Conclusion

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Thank you !