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Charge Quantization in Phase Space

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OUTLINE

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Introduction:

-Wigner's function , since 1932, is the most commonly used phase-space distribution function for physical systems

$$\boxed{W(p,q)=\frac{1}{2\pi\hbar}\int dy\,\psi^*(q-y/2)\psi(q+y/2)e^{-ipy/\hbar}}$$

Real

Negative

All distribution functions are normalized.

$$\int W(p,q)\,dpdq=1$$

where the corresponding dynamical equation is

$$i\hbar \frac{\partial W}{\partial t} = -i\hbar \frac{p}{m} \frac{\partial}{\partial q} W + \sum_{n-odd} \frac{(i\hbar/2)^n}{n!} \frac{\partial^n V}{\partial q^n} \frac{\partial^n}{\partial p^n} W$$

The averaging rule is

$$\left\langle \hat{O}(\hat{p},\hat{q}) \right\rangle = \int O(p,q) W(p,q) dpdq$$

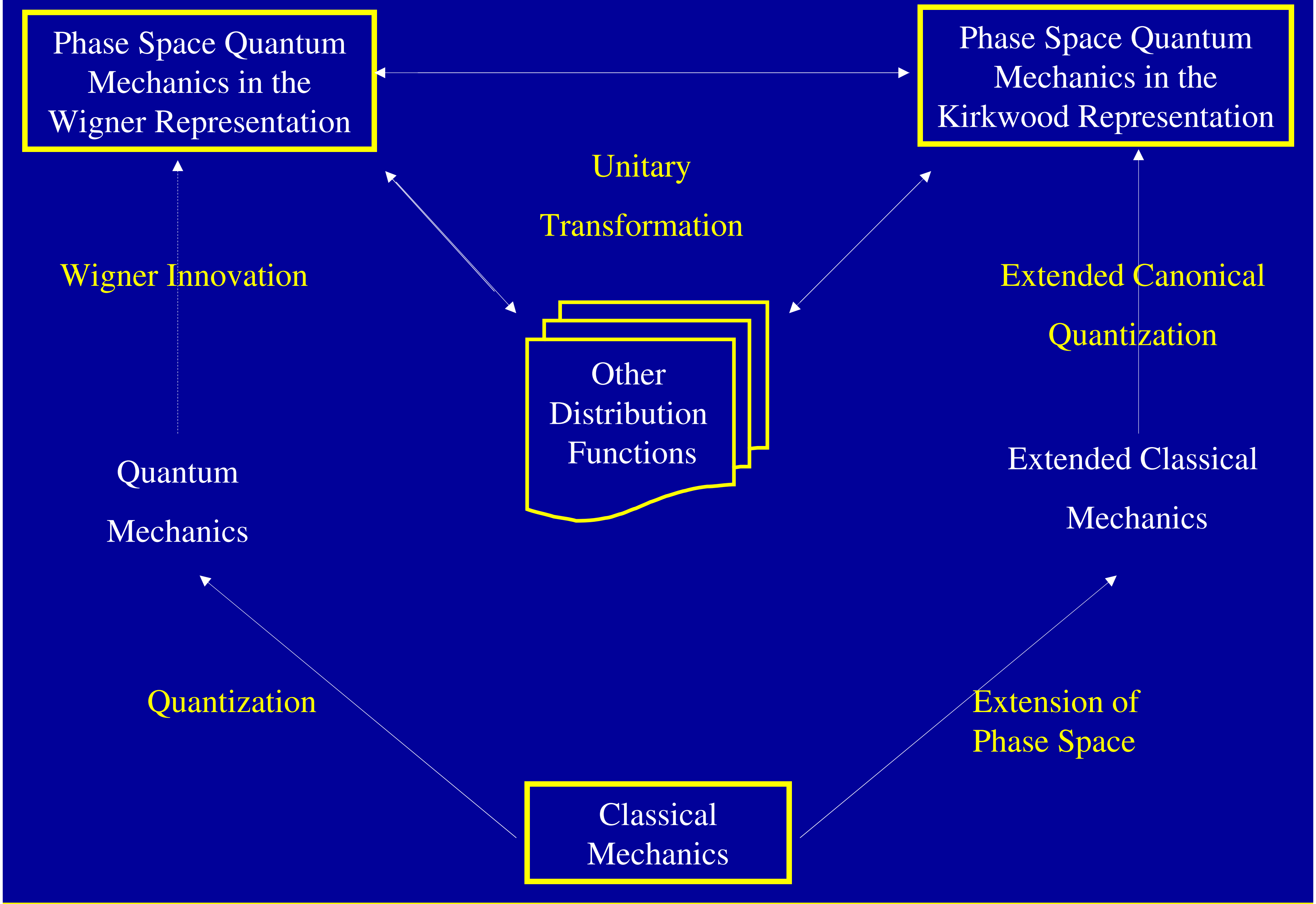
Where the c-functions corresponding to the quantum mechanical operators are obtained from Weyl-Wigner transformation. [Wigner, 1932]

- ❑"Ordering rule" is a key word in the phase space quantum mechanics.
  - ❑ Each distribution function has its ordering rule.
    - ❑Wigner DF has Wyle ordering rule.

$$In \qquad General \qquad (\hat{p})^n (\hat{q})^m \neq (p)^n (q)^m$$

Distribution functions in chronological order

Wigner Distribution Function	Wigner	1932	Wyle Ordering
Kirkwood Distribution Function	Kirkwood	1933	Anti-Standard
Husimi Distribution Function	Husimi	1940	Anti-Normal
Glauber-Sudarshan Distribution Function (or P-Distribution Function)	Glauber &Sudarshan	1963 1963	Normal
Mehta Distribution Function	Mehta	1964	Standard
Q-Distribution Function	Glauber	1967	Anti-Normal
Other……………….	-----------------		---------------



Phase Space Quantum Mechanics in the Wigner Representation

Phase Space Quantum Mechanics in the Kirkwood Representation

Unitary

Transformation

Wigner Innovation

Extended Canonical

Quantization

Other Distribution Functions

Quantum

Mechanics

Extended Classical

Mechanics

Quantization

Extension of Phase Space

Classical Mechanics

Extension of the Classical Lagrangian and Hamiltonian

The ordinary Lagrangian is a function of generalized coordinates q, velocity $\dot{q}$.

$$L^q = L(q,\dot{q}) \qquad q = \{q_i(t), \qquad i = 1,2,\dots,N\}$$

Using the Legendre transformation, for a given functional form of Hamiltonian H*(p,q)*, the following equation may be considered as a differential equation for L^q.

$$\mathrm{H}\,(q\,,\frac{\partial L^q}{\partial \dot{q}}) = \dot{q}\,\frac{\partial L^q}{\partial \dot{q}} - L^q\,.$$

Let L^p be a lagrangian in p representation. (See Goldstein, 1980, p. 372, for this type of lagrangians)

$$L^p = L(p,\dot{p}) \qquad p = \{p_i(t), \qquad i = 1,2,\dots,N\}$$

It is related to H*(p,q)* as follows

$$\mathrm{H}\,(\,p\,,\frac{\partial L^p}{\partial \dot{p}}) = \dot{p}\,\frac{\partial L^p}{\partial \dot{p}} - L^p\,.$$

The correct classical dynamical equation of the system may be obtained from both Lagrangians using the corresponding Euler-Lagrange's equations.
[Sobouti&Nasiri 1993, Nasiri et al. 2006]

One may combine the two pictures and define an *extended* lagrangian in the phase space as the sum of q and p lagrangians

$$L(\dot{q}, q, \dot{p}, p) = -\frac{d(pq)}{dt} + L^{q} + L^{p}$$

It is easy to show that this extended Lagrangian gives the conventional classical equations through the following Euler-Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}} - \frac{\partial L}{\partial \eta} = \frac{d}{dt}\frac{\partial L^{\eta}}{\partial \dot{\eta}} - \frac{\partial L^{\eta}}{\partial \eta} = 0. \qquad \eta \equiv \{q \quad or \quad p\}.$$

We treat q and p as independent variables in phase space. Thus one may define their conjugate momenta as

$$\pi_q = \frac{\partial L}{\partial \dot{q}}, \quad \pi_p = \frac{\partial L}{\partial \dot{p}}.$$

$$(p,q) \xrightarrow{Extension} (p,q,\pi_p,\pi_q)$$

By definition of extended lagrangian on has

$$\pi_q = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L^{(q)}}{\partial \dot{q}} - p, \quad \pi_p = \frac{\partial L}{\partial \dot{p}} = \frac{\partial L^{(p)}}{\partial \dot{p}} - q.$$

whenever the extended conjugate momenta vanish these reduce to the conventional definitions for q and p.

By Legendre's transformation one may define the Extended Hamiltonian as

$$\mathbb{H}(p,q,\pi_p,\pi_q) = \dot{q}\pi_q + \dot{p}\pi_p - L(p,q,\dot{p},\dot{q},t) = \mathrm{H}(\mathrm{q},\mathrm{p}+\pi_{\mathrm{q}}) - \mathrm{H}(q+\pi_p, p)$$

Extended Canonical Quantization

One may define the extended poisson's brackets as

$$\{\{A,B\}\}=\frac{\partial A}{\partial q}\frac{\partial B}{\partial \pi_q}-\frac{\partial A}{\partial \pi_q}\frac{\partial B}{\partial q}+\frac{\partial A}{\partial p}\frac{\partial B}{\partial \pi_p}-\frac{\partial A}{\partial \pi_p}\frac{\partial B}{\partial p}.$$

Let $\chi(p,q)$ belong s to a complex function space and $p,\ q,\ \pi_p\ and\ \pi_q$ are operators in this space which satisfy the following commutation relations

$$[q,\pi_q]=[p,\pi_p]=i\hbar,\quad Others\equiv 0.$$

These are the extended Poisson's brackets which are promoted to the commutation relations by Dirac's prescription, i.e.

$$\{\{\ \ ,\ \ \}\}\rightarrow\frac{1}{i\hbar}[\ \ ,\ \ ].$$

The dynamical equation is given by

$$\mathbb{H}(p,q,\pi_p,\pi_q)\chi(p,q,t)=i\hbar\frac{\partial}{\partial t}\chi(p,q,t).$$

where

$$\mathbb{H}(p,q,\pi_p,\pi_q)=\mathrm{H}(\mathrm{q},\mathrm{p}+\pi_{\mathrm{q}})-\mathrm{H}(q+\pi_p,p)$$

The expectation values are obtained by the following rule

$$Tr(\hat{O}\hat{\rho})=\langle O(p,q)\rangle=\int O(p,q)\chi^* dpdq,\qquad \hat{O}(\hat{p},\hat{q})\xrightarrow{\text{Anti}-\text{Standard}-\text{Ordering}}O(p,q)$$

A general solution for the dynamical equation is as

$$\chi(p,q,t)=\sum_{\alpha,\beta}A_{\alpha\beta}\psi_{\alpha}(q,t)\varphi_{\beta}^{*}(p,t)e^{-ipq/\hbar}.$$

For a pure state one has $\chi(p,q,t)=\psi(q,t)\varphi^{*}(p,t)e^{-ipq/\hbar}$. This is the well-known Kirkwood distribution function and obeys the anti-standard ordering rule.

The Wigner distribution function (and other distribution functions in phase space) would be obtained from that of the Kirkwood by a unitary transformation: [Sobouti&Nasiri 1993] , [Hai-Woong Lee 1995]

$$\chi=\exp(-\frac{i}{2\hbar}\pi_{p}\pi_{q})W.$$

Gauge Transformation in Phase Space

In the ordinary quantum mechanics the gauge transformations (GT) are defined by a unitary transformation as

$$U = \exp(-\tfrac{iQ}{\hbar c}\Lambda(q,t))$$

Where Λ is an arbitrary Hermitian function of coordinates and time and Q is the electric charge. The wave functions are transformed by the GT

$$\psi' = U\psi.$$

The Schrödinger equation is invariant under the GT.

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \xrightarrow{GT} H'\psi' = i\hbar\frac{\partial\psi'}{\partial t}, \qquad H' = UHU^{+} - i\tfrac{\hbar c}{Q}U(\frac{\partial U^{+}}{\partial t}).$$

The effect of GT on the Wigner distribution function is obtained as

$$\begin{aligned} W'(p,q) &= \tfrac{1}{2\pi\hbar}\int dy\,\psi'^{*}(q-y/2)\psi'(q+y/2)e^{-ipy/\hbar}, \\ &= \tfrac{1}{2\pi\hbar}\int dy\exp(\tfrac{iQ}{\hbar c}\Lambda(q-y/2))\psi^{*}(q-y/2)\exp(\tfrac{-iQ}{\hbar c}\Lambda(q+y/2))\psi(q+y/2)e^{-ipy/\hbar} \\ &= \tfrac{1}{2\pi\hbar}\int dy\exp(\tfrac{-iQ}{\hbar c}\{\int_{-y/2}^{y/2}\nabla\Lambda(q+s)ds\})\psi^{*}(q-y/2)\psi(q+y/2)e^{-ipy/\hbar} \\ &= \tfrac{1}{2\pi\hbar}\int dy\,\psi^{*}(q-y/2)\psi(q+y/2)\exp(-\tfrac{i}{\hbar}\{\int_{-y/2}^{y/2}[p+\tfrac{Q}{c}A'(q+s)]ds\}) \end{aligned}$$

which is the Wigner distribution function for a particle interacting with the EM fields.

It is easy to show that if the Wigner distribution function be normalized the gauge transformed Wigner distribution function is also normalized

$$\int dqdpW(p,q) = \int dq\,|\psi(q,t)|^2 = \int dp\,|\varphi(p)|^2 = 1,$$

$$\int dqdpW'(p,q) = \int dq\,|\psi'(q,t)|^2 = \int dp\,|\varphi'(p)|^2 = 1.$$

- One may define the GT of the Kirkwood distribution function

$$\chi' = \psi'\varphi'^{*} e^{-ipq/\hbar} = \exp(-\frac{iQ}{\hbar c}\{\Lambda(q,t) - \Gamma(p,t)\})\chi \qquad \textbf{(A)}$$

$$= \exp(-\frac{iQ}{\hbar c} g_0(p,q,t))\chi = G_O(p,q)\chi,$$

Where G_o is a unitary GT in phase space, corresponding to the ordinary GT [Khademi&Nasiri, 2002]

To derive the gauge transformed Kirkwood distribution function (A) one may use the GT in the p-space quantum mechanics, where

$$\varphi'(p) = S\varphi(p), \qquad S = \exp(\frac{-iQ}{\hbar c}\Gamma(p,t))$$

The arbitrary and Hermitian gauge function in p-space Γ is related to the well known gauge function in the q-space Λ by

$$\Gamma(p,t) = -\frac{i\hbar c}{Q}\ln\left\{\frac{\int dq\,\exp(\frac{iQ}{\hbar c}\Lambda(q))\psi^{*}(q,t)e^{-ipq/\hbar}}{\int dq\,\psi^{*}(q,t)e^{-ip/\hbar}}\right\}$$

One can rewrite Eq. (A) as

$$\chi' = \frac{1}{2\pi\hbar}\int dy\ \psi'(q)\psi'^{*}(q+y)e^{-ipy/\hbar},$$

$$= \frac{1}{2\pi\hbar}\int dy\ \exp(-\frac{iQ}{\hbar c}\{\Lambda(q)-\Lambda(q+y)\}\psi(q)\psi^{*}(q+y)e^{-ipy/\hbar},$$

$$= \frac{1}{2\pi\hbar}\int dy\ \exp(-\frac{iQ}{\hbar c}\{\int_{0}^{y}\nabla\Lambda(s)ds\}\psi(q)\psi^{*}(q+y)e^{-ipy/\hbar},$$

$$= \frac{1}{2\pi\hbar}\int dy\ \psi(q)\psi^{*}(q+y)\exp(-\frac{i}{\hbar}\{\int_{0}^{y}[p+\frac{Q}{c}A'(q+s)]ds\})$$

This is the Kirkwood distribution function for a particle interacting with the EM fields.

Compare χ' and W'

$$W' = \frac{1}{2\pi\hbar}\int dy\psi^{*}(q-y/2)\psi(q+y/2)\exp(-\frac{i}{\hbar}\{\int_{-y/2}^{y/2}[p+\frac{Q}{c}A'(q+s)]ds\})$$

General Gauge Transformations

The ordinary GT in phase space are obtained by

$$G_O = \exp(-\frac{iQ}{\hbar c} g_O), \qquad g_O = \Lambda(q,t) - \Gamma(p,t).$$

The gauge invariance of dynamical equation in the phase space yields

$$\mathbb{H}\chi = i\hbar\partial\chi/\partial t \xrightarrow{GT} \mathbb{H}'\chi' = i\hbar\partial\chi'/\partial t,$$

$$\chi' = G_O\chi, \qquad \mathbb{H}' = G_O \mathbb{H} G_O^{+}, \quad time \quad independent\ gauges$$

Clearly, for the local gauges that the gauge function assumes a constant value, the distribution functions do not change

$$\Gamma(p,t) = -\frac{i\hbar c}{Q} \ln\left\{\frac{\int dq\ \exp(\frac{iQ}{\hbar c}\Lambda(q))\psi^*(q,t)e^{-ipq/\hbar}}{\int dq\ \psi^*(q,t)e^{-ip/\hbar}}\right\}$$

$$if \quad \Lambda = cons. \Rightarrow \Gamma = \Lambda = cons. \Rightarrow g_O = 0 \Rightarrow \chi' = \chi$$

Suppose that the ordinary gauges in phase space are a subset of a more general gauge in phase space , denoted by $g = g(p,q)$.

It is evident that if $g_O(p,q) = \Lambda(q) - \Gamma(p) \in \mathbb{C}_0$, and $g(p,q) \in \mathbb{C}$, then, $\mathbb{C}_0 \subset \mathbb{C}$.

$$G = \exp(\frac{iQ}{\hbar c} g(p,q,t))$$

Question: Is the dynamical equation in the phase space invariant under the general gauge transformation?

Let us first consider the general global gauge transformation: $g = cons.$

The dynamical equations in phase space are invariant under global gauge transformation,

however, the normalization condition of the corresponding distribution functions are violated

$$if \quad \int dpdq\chi^* = 1 \xrightarrow{GT} \int dpdq\chi'^* = e^{\frac{iQ}{\hbar c}g}\int dpdq\chi^* = e^{\frac{iQ}{\hbar c}g}.$$

As an except ion, when $e^{\frac{iQ}{\hbar c}g} = e^{2in\pi}$, $n = \{0, \pm 1, \pm 2, \ldots\}$ which yields a **discrete global gauge symmetry in the phase space,** the normalization condition is still satisfied.

Now consider a local gauge transformation, then , in general, one has a constraint on the local gauges that leaves them to be discrete

$$\int e^{iQg(p,q)/\hbar c}\chi^* dpdq = < e^{iQg(p,q)/\hbar c} > = e^{i2n\pi}, \quad n = 0,1,2,\ldots,k,\ldots,$$

Charge Quantization in the Phase Space

The discrete gauge symmetry is shown by following constraint

$$< e^{iQg(p,q)/\hbar c} >= e^{i2n\pi} , \quad n = 0,\pm1,\pm2,\dots,$$

One may expand both sides of the above equation to get

$$< \sum_m \frac{(iQ/\hbar c)^m}{m!} g(p,q)^m >= \sum_m \frac{(i)^m}{m!}(2\pi n)^m \Rightarrow \sum_m \frac{(i)^m}{m!}\left((Q/\hbar c)^m < g(p,q)^m > -(2\pi n)^m\right) = 0, \quad 0 \le m \le \infty.$$

The above equation means that the gauge function $g(p,q)$ is not really an arbitrary one. In other words, there is still some arbitrariness in choosing $g(p,q)$. The gauge functions that obeys this constraint condition, belong to a set of constraint gauge functions

$$g(p,q) \in \{g(p,q)\} \equiv \mathbb{C}_c$$

where the index c indicates the constraint. Due to the arbitrariness of the gauge functions, one may choose the gauge functions as the member of a subset of $\mathbb{C}_c$

$$g^{(k)}(p,q) \in \{g^{(k)}(p,q)\} \equiv \mathbb{C}_c^k \subset \mathbb{C}_c$$

for which at least a single term in the summation vanishes. For example, if one chooses a gauge function for which the kth term of the summation vanishes , then

$$\left((Q/\hbar c)^k < g(p,q)^k > -(2\pi n)^k\right) = 0,$$

$$Q(\sqrt[k]{< g(p,q)^k >}) = 2\pi n\hbar c \qquad \textbf{(a)}$$

Equation (a) shows the quantization of the charge Q. For n=1 one has the quanta of charge, Q=e. Thus

$$\sqrt[k]{< g(p,q)^k >} = 2\pi\hbar c / e \qquad \textbf{(b)}$$

Substituting Eq. (b) in (a) gives

$$\boxed{Q = ne.}$$

Dirac pioneered in this direction and argued that in the presence of magnetic monopole with the magnetic charge Q_m, the electric charge is quantized as

$$QQ_m = 2\pi n\hbar c$$

Dirac assumes:

1-the Dual symmetry for Maxwell's equation.

2-existence of magnetic monopoles.

3-a singular vector potential, to obtain the correct magnetic field due to a magnetic monopole.

But the magnetic monopole is not detected yet!

If the magnetic charge could never be detected, what happens to the charge quantization which has a strong experimental verification?

To obtain the quantization of charge we assume:

1-the discrete gauge symmetry in the phase space quantum mechanics which guarantees the normalization condition.

2-nothing

It is important to note that the **conservation of charge** is obtained as a result of the gauge symmetry in both configuration and phase space picture of QM, while **the quantization of charge is the property of the discrete gauge symmetry in phase space formulation of QM**.

Charge Quantization in the Wigner Representation

The Kirkwood distribution function is a complex function, while that of the Wigner, W(p,q) , is a real one. A general gauge transformation transforms the Wigner function into a new distribution function which is complex. Therefore

“in general an extended gauge transformation may change the representation in the phase space”

Now, we restrict ourselves to those gauges which keeps the Wigner representation unchanged. Thus, the only gauge symmetry of the Wigner representation is the discrete global gauges symmetry.

$$W' = \exp(-iQg/\hbar c)W = e^{-2in\pi}W,$$

then

$$Qg/\hbar c = 2n\pi, \qquad \text{(c)}$$

where g is a (global) gauge. Equation (c) not only keeps the reality property of a distribution function, but also the appearance of the factor 2 in that equation, guarantees the invariance of normalization condition under the discrete gauge transformations

$$\int W(p,q)dpdq = 1.$$

By setting n=1 in Eq. (c) one gets

$$g = g_1 = 2\pi\hbar c / e$$

then one obtains the charge quantization in the Wigner representation

$$\boxed{Q = ne.}$$

Another Property:

It is shown that the quantization of extended action

$$S_{Extended}\,(p,q) = \int L(\dot{p},\dot{q},p,q,t)dt = n\pi\hbar.$$

in phase space could be obtained as a results of the reality of the Wigner function, that immediately gives the quantization of the energy. [Nasiri 2007].

The extended action changes under an extended gauge transformation as

$$L' = L + \tfrac{Q}{c}\frac{d}{dt}g \Rightarrow S' = S + \tfrac{Q}{c}g = S + n\tfrac{e}{c}g$$

Thus, one may argue that the quantization rule of the extended action is invariant under the discrete gauge transformations in the Wigner representation of QM in phase space.

Conclusions

- A systematic approach for introducing the Kirkwood and Wigner distribution functions in terms of the extended Lagrangian and Hamiltonian followed by the extended canonical quantization is presented.

- As well as ordinary gauge transformations in the phase space quantum mechanics, a discrete (local and global) gauge symmetry for the dynamical equations in the phase space is introduced. This gauges had to be discrete due to the conservation of normalization condition for the distribution functions in the phase space.

- Charge quantization is obtained for a complex (Kirkwood) distribution function as an immediate consequence of the discrete gauge symmetry in the phase space.

- Charge quantization is obtained for the Wigner representation of QM in phase space, as well.

- It is shown that the quantization of extended action is invariant under this discrete symmetry.

- This approach to the charge quantization does not need the concepts of magnetic monopole, dual symmetry of Maxwell's equations, and singular vector potentials.

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Thanks a lot