QUANTUM TUNNELING OF BOSONS

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1 Motivation

- Classical dynamics - described by Lagrangian or Hamiltonian mechanics in a phase space; works well at macroscopic scales.
- Quantum dynamics - described by Schrödinger's equation in a Hilbert space; works well at microscopic scales.
- The microscopic/macroscopic interface between the two is not well understood. However the interface is becoming increasingly important from an engineering and technology perspective.
- There are features of the quantum theory which have no analogue in the classical theory. Examples are the notions of *indistinguishable particles*, *Schrödinger cat states* and *tunneling*. These aspects are relevant in the theory of Bose–Einstein condensates (BECs).
- Promising progress in experimental BEC techniques gives hope that *controlable macroscopic quantum systems* will be realised in the laboratory, with the potential to clarify the nature of the microscopic/macroscopic interface.

- BECs were first predicted in 1925. Loosely, a BEC is a system where a significant fraction of the particles occupy the same quantum state. It is achieved by cooling a system below a critical temperature. For the non-interacting spinless boson case

$$T_c = \left(\frac{n}{\zeta(3/2)} \right)^{2/3} \frac{h^2}{2\pi m k_B}.$$

- In 1938 Kapitsa, Allen, Misener cooled ^{4}He to 2.17K and yielded a (strongly interacting) superfluid state.

- In 1995 Cornell and Wieman lead the creation of the first BEC of about 2000 (weakly interacting) ^{87}Rb atoms at $\sim 10^2$ nK.

- Later in the same year a group lead by Ketterle produced a BEC with $\sim 10^5$ atoms of ^{23}Na.

- Atomic interactions may be either attractive (e.g. ^{85}Rb) or repulsive (e.g. ^{85}Rb, ^{87}Rb). Densities are typically 10^{-5} that of air.

- There has been a wealth of subsequent activity such as *Bosenova* phenomena, molecular condensates, and condensates of fermionic Cooper pairs.

2 Non-interacting bosons

Consider the Hamiltonian for a single particle in a harmonic potential:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}$$

which is well known to have energy levels $E_n = \hbar\omega\,(n + 1/2)$. For a system of N non-interacting bosons in such a potential the total energy is

$$E = \sum_n N_n E_n \qquad \text{such that} \qquad \sum_n N_n = N.$$

Theorem 1 The Hellmann-Feynman theorem*:*
For a given Hamiltonian H which smoothly depends on a coupling parameter γ define

$$\mathcal{A} = \frac{\partial H}{\partial \gamma}.$$

For every normalised eigenstate $|\psi_n\rangle$ of H with energy E_n there holds

$$\langle\psi_n|\,\mathcal{A}\,|\psi_n\rangle = \frac{\partial E_n}{\partial \gamma}.$$

From the Hellmann-Feynman theorem we find the following identity

$$E_n = \frac{1}{2m} \langle \psi_n | p^2 | \psi_n \rangle + \frac{m\omega^2}{2} \langle \psi_n | x^2 | \psi_n \rangle$$

with

$$\langle \psi_n | p^2 | \psi_n \rangle = \frac{\hbar m\omega(2n+1)}{2}, \qquad \langle \psi_n | x^2 | \psi_n \rangle = \frac{\hbar(2n+1)}{2m\omega}.$$

By symmetry arguments we also have

$$\langle \psi_n | p | \psi_n \rangle = 0, \qquad \langle \psi_n | x | \psi_n \rangle = 0.$$

A more general case may be considered using a quartic potential

$$V(x) = V_0 - 2V_1 x^2 + V_2 x^4$$

with $V_2 > 0$. By symmetry we again have

$$\langle \psi_n | p | \psi_n \rangle = 0, \qquad \langle \psi_n | x | \psi_n \rangle = 0.$$

The case cannot be solved exactly so we appeal to a classical (or *adiabatic*) approximation.

By ignoring the kinetic energy, the classical ground-state energy $\tilde{E}_0$ is given by

$$\tilde{E}_0 \sim \begin{cases} V_0 & \text{for } V_1 \leq 0 \\ V_0 - V_1^2/V_2 & \text{for } V_1 \geq 0. \end{cases} \tag{1}$$

Using this result as an approximation to the ground-state energy for the quantum system, we can appeal to the Hellmann–Feynman theorem to approximate the ground-state position fluctuations:

$$\langle\psi_0| x^2 |\psi_0\rangle = -\frac{1}{2}\frac{\partial \tilde{E}_0}{\partial V_1} \sim \begin{cases} 0 & \text{for } V_1 \leq 0 \\ V_1/V_2 & \text{for } V_1 \geq 0. \end{cases}$$

In a full quantum analysis we must expect that quantum fluctuations will smooth out the discontinuity in the derivative of $\langle\psi_0| x^2 |\psi_0\rangle$ (and in particular $\langle\psi_0| x^2 |\psi_0\rangle > 0$ for all V_1), so there is no *phase transition* in the traditional sense. Nonetheless, the above classical analysis indicates that around the crossover coupling $V_1 = 0$ we should expect a noticible change in the behaviour of $\langle\psi_0| x^2 |\psi_0\rangle$. This change is associated with a *bifurcation* of the potential.

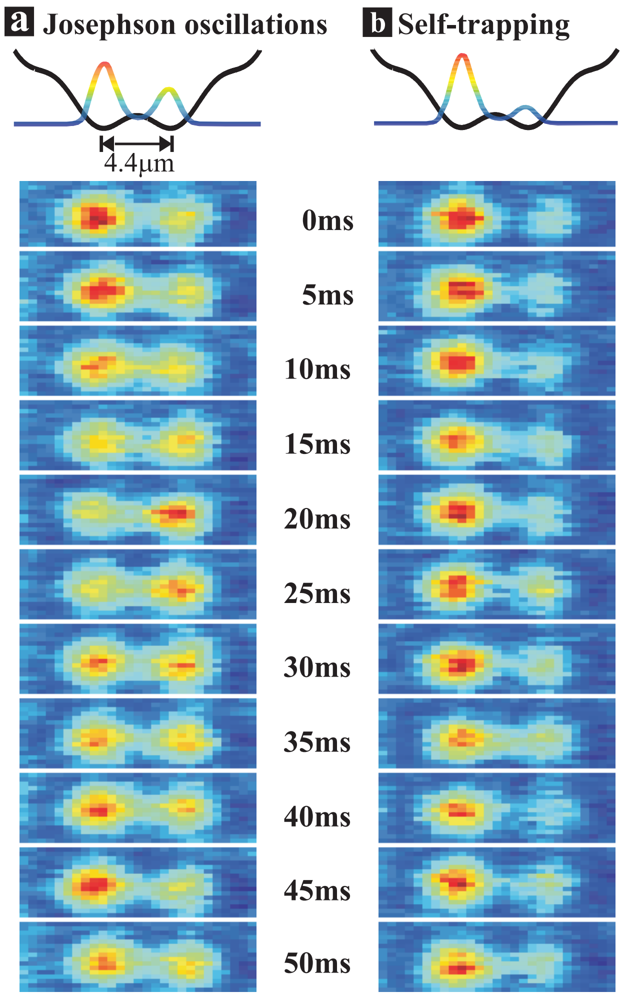

Figure 1: Figure reproduced from *Direct observation of tunneling and nonlinear self-trapping in a single bosonic Josephson junction*, M. Albiez, R. Gati, J. Fölling, S, Hunsman, M. Cristiani and M. Oberthaler, Physical Review Letters **95** (2005) 010402.

"We are able to measure the population of both spin states nondestructively using phase-contrast microscopy [14,15]. We tune the probe laser between the resonant optical frequencies for the $|1\rangle$ and $|2\rangle$ states. Since the probe detuning has an opposite sign for the two states, the resulting phase shift imposed on the probe light has an opposite sign, such that the $|1\rangle$ atoms appear white and the $|2\rangle$ atoms appear black against a gray background on the CCD array. We can acquire multiple, nondestructive images of the spatial distribution of the $|1\rangle$ and $|2\rangle$ atoms at various discrete moments in time, or we can acquire a quasicontinuous time record (streak image) of the difference of the populations in the $|1\rangle$ and $|2\rangle$ states, integrated across the spatial extent of the cloud."

Quote reproduced from *Watching a Superfluid Untwist Itself: Recurrence of Rabi Oscillations in a Bose-Einstein Condensate*, M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, M. J. Holland, J. E. Williams, C. E. Wieman, and E. A. Cornell, Physical Review Letters **83** (1999) 3358.

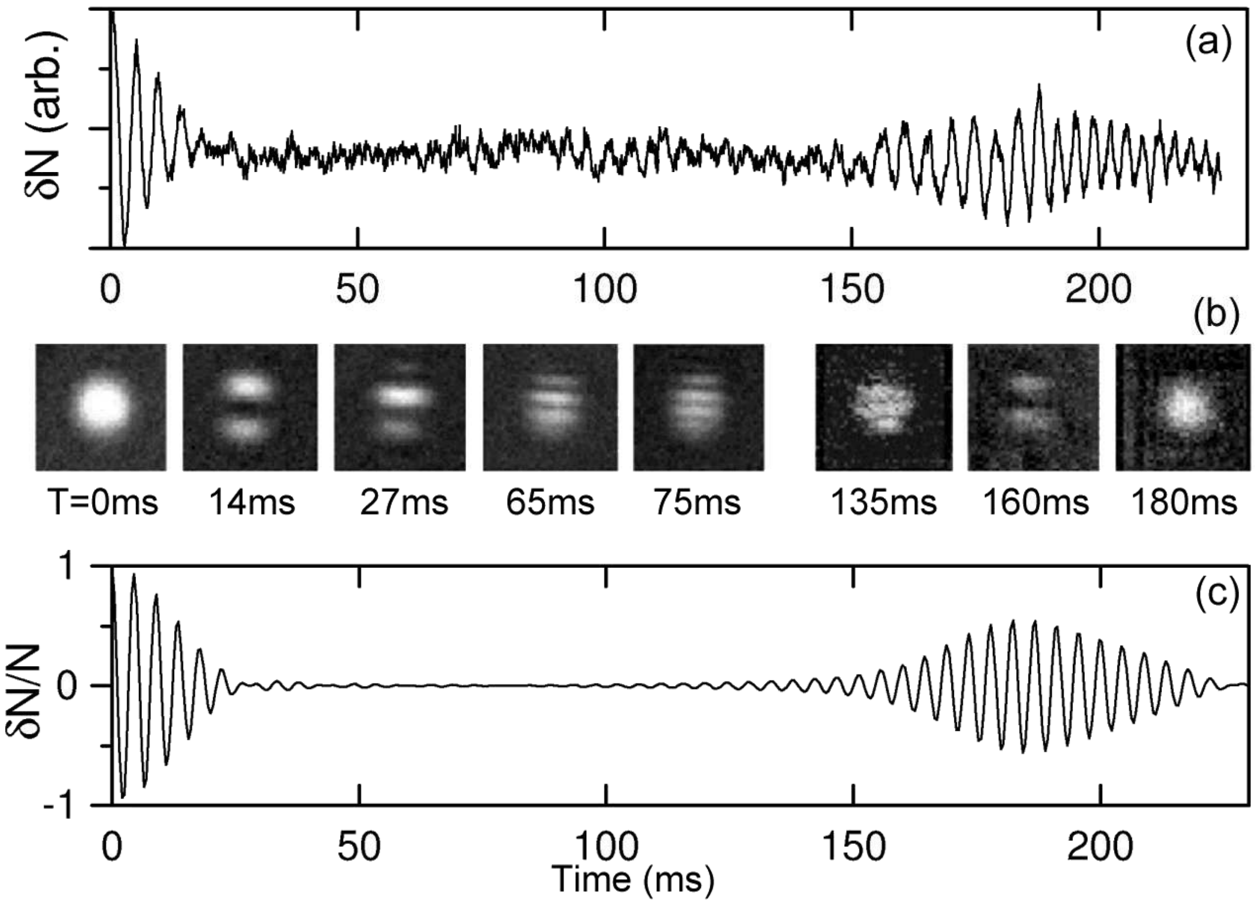

Figure 2: Figure reproduced from *Watching a Superfluid Untwist Itself: Recurrence of Rabi Oscillations in a Bose-Einstein Condensate*, M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, M. J. Holland, J. E. Williams, C. E. Wieman, and E. A. Cornell, Physical Review Letters **83** (1999) 3358.

3 An effective model for quantum tunneling

- Self-trapping phenomena have been studied in the context of non-linear dynamics since the mid 1970s.
- Quantisation of such systems provide candidates for the study of quantum self-trapping phenomena.
- For tunneling of bosons between two wells, we will take the simplifying assumption that the effective system may be treated with just two degrees of freedom; i.e. one degree of freedom for each well. The Hilbert space is then spanned by the basis

$$|m, n\rangle = (b_1^\dagger)^m (b_2^\dagger)^n |0\rangle$$

where

$$[b_j,\, b_k] = [b_j^\dagger,\, b_k^\dagger] = 0, \qquad [b_j,\, b_k^\dagger] = \delta_{jk} I \tag{2}$$

and $N_1 = b_1^\dagger b_1$ and $N_2 = b_2^\dagger b_2$.

Quantising the self-trapping equations leads to

$$H = \frac{k}{8}(N_1 - N_2)^2 - \frac{\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}}{2}(b_1^\dagger b_2 + b_2^\dagger b_1)$$

- k - scattering interaction strength
- μ - external potential
- $\mathcal{E}$ - tunneling strength

The change $\mathcal{E} \to -\mathcal{E}$ corresponds to the unitary transformation $b_1 \to b_1$, $b_2 \to -b_2$, while $\mu \to -\mu$ corresponds to $b_1 \leftrightarrow b_2$. The quantity $N = N_1 + N_2$ is the total boson number operator and is conserved; i.e. $[H, N] = 0$. Since the system has two degrees of freedom and there are two conserved operators, the system is integrable.

The same Hamiltonian is obtained in a minimal mode approximation of an appropriate quantum field theory of coupled *non-linear* Schrödinger equations.

$$H = \frac{k}{8}(N_1 - N_2)^2 - \frac{\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}}{2}(b_1^\dagger b_2 + b_2^\dagger b_1)$$

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