Solvable Critical Dense Polymers

by

Jørgen Rasmussen

based on

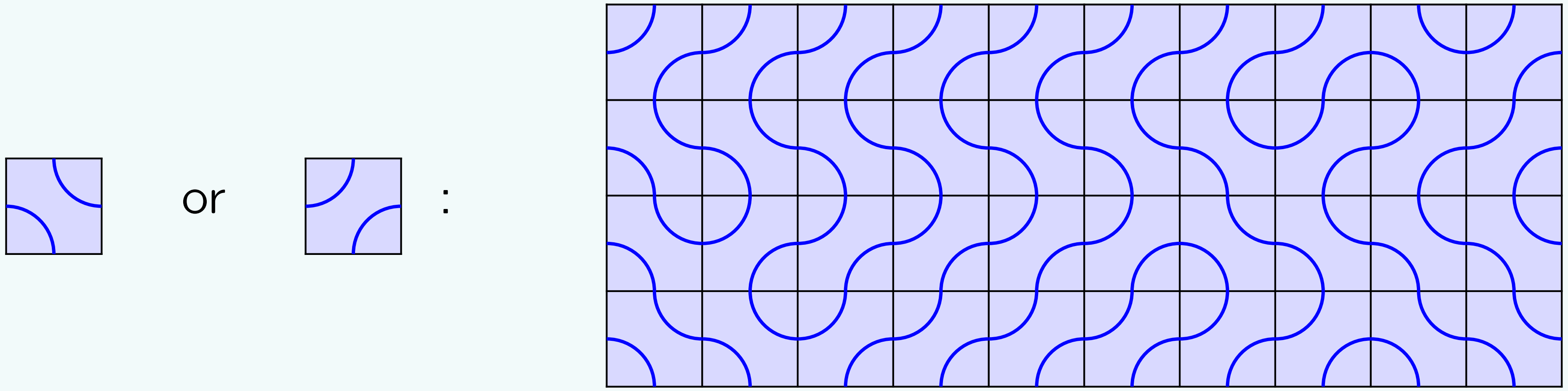
- P.A. Pearce, J. Rasmussen, J.-B. Zuber, *Logarithmic Minimal Models*, J. Stat. Mech. 0611 (2006) P017
- P.A. Pearce, J. Rasmussen, *Solvable Critical Dense Polymers*, hep-th/0610273
- P.A. Pearce, J. Rasmussen, *Physical Combinatorics of Critical Dense Polymers* (2007)

Outline

- Lattice model
- Planar Temperley-Lieb algebra
- Double-row transfer matrix and link states
- Inversion identity
- Finite-size corrections
- Physical combinatorics: selection rules and characters
- Relation to Dyck paths
- Hamiltonian limits
- Fusion and Jordan cells

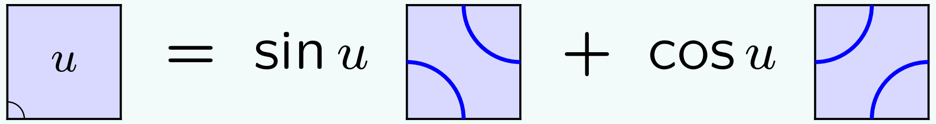
Lattice Model and Planar TL Algebra

Typical Configuration —dense loop representation—



—with some boundary conditions

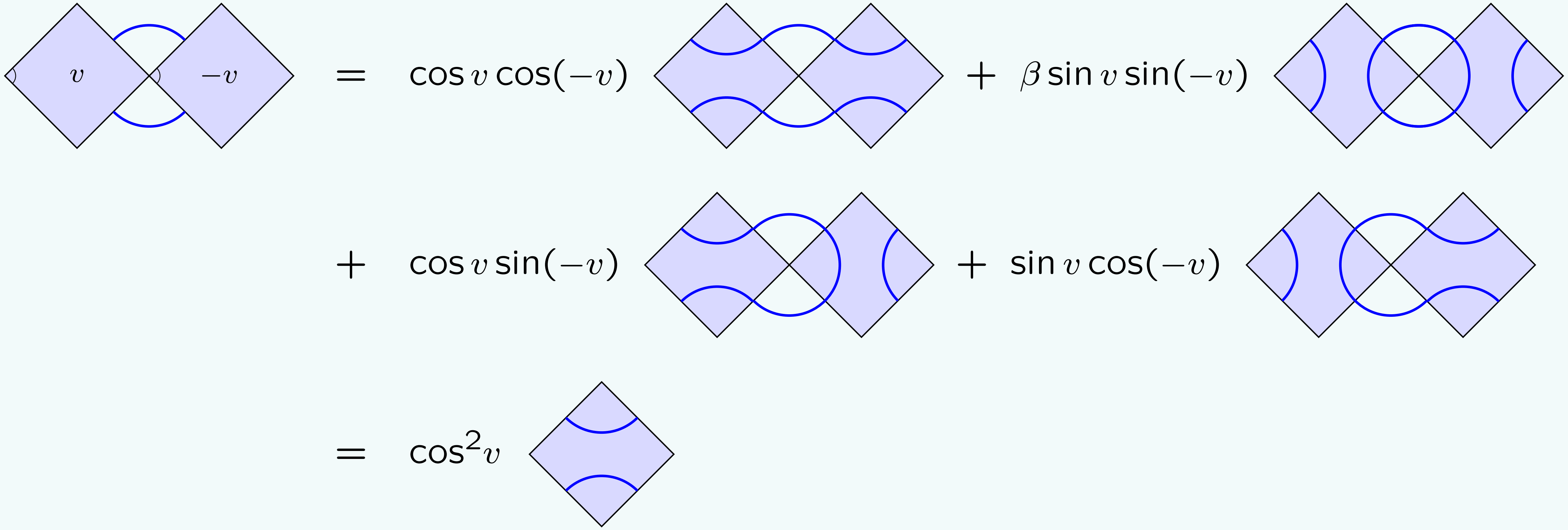
Face Operators



- Spectral parameter: u
- Crossing parameter: $\lambda = \frac{\pi}{2}$
- Fugacity: $\beta = 2\cos\lambda = 0$ ⟶ **NO LOOPS**

Local Properties

Inversion Relation

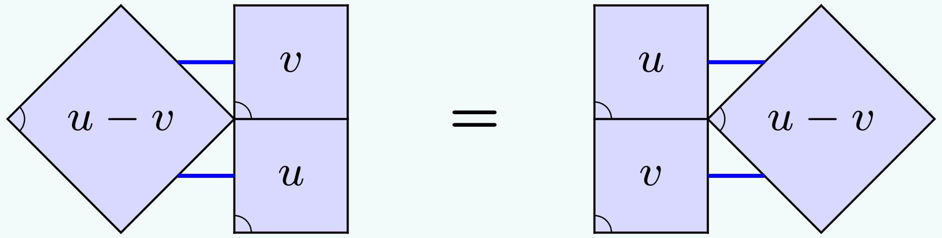


$= \cos v \cos(-v)$ $+\ \beta \sin v \sin(-v)$

$+\ \cos v \sin(-v)$ $+\ \sin v \cos(-v)$

$= \cos^2 v$

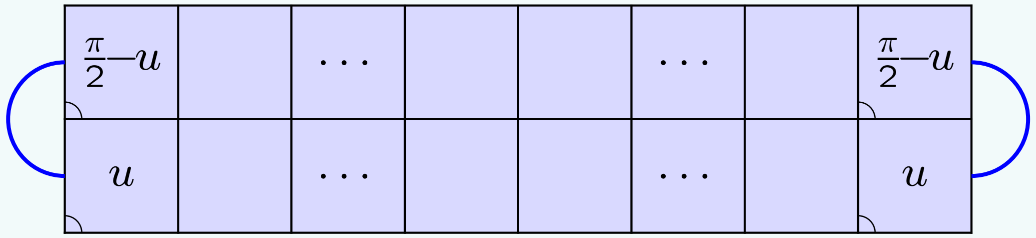
Yang-Baxter Equation



Double-Row Transfer Matrix

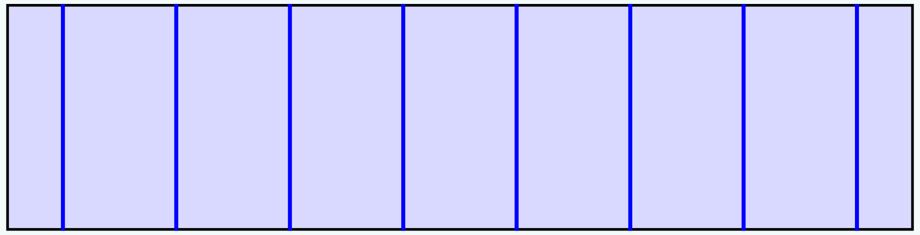
Definition as N-tangle

$$\boldsymbol{D}(u) \;=\; \frac{1}{\sin 2u}$$



Normalization

$$\lim_{u\to 0}\boldsymbol{D}(u) \;=\; \boldsymbol{I} \;=$$



Commuting Family

$$\boxed{\boldsymbol{D}(u)\boldsymbol{D}(v) = \boldsymbol{D}(v)\boldsymbol{D}(u)}$$

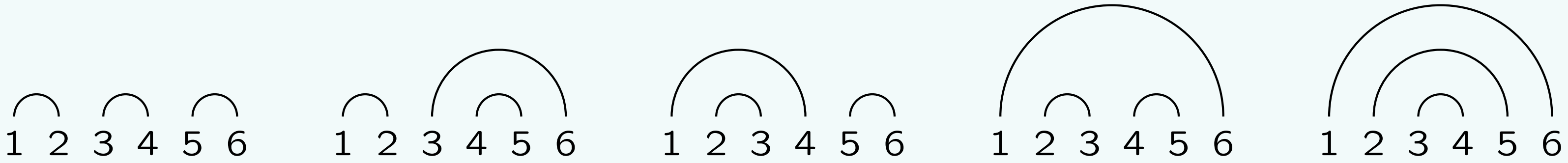
- Multiplication is vertical concatenation of diagrams
- Equality is the equality of N-tangles

Crossing Symmetry

$$\boxed{\boldsymbol{D}(u) = \boldsymbol{D}(\lambda - u)}$$

Link States

- In a ‘fixed direction’, the planar N-tangles can act on a vector space of link diagrams.
- Number of nodes: N
- For $N = 6$ and no defects, there is a basis of 5 link states:

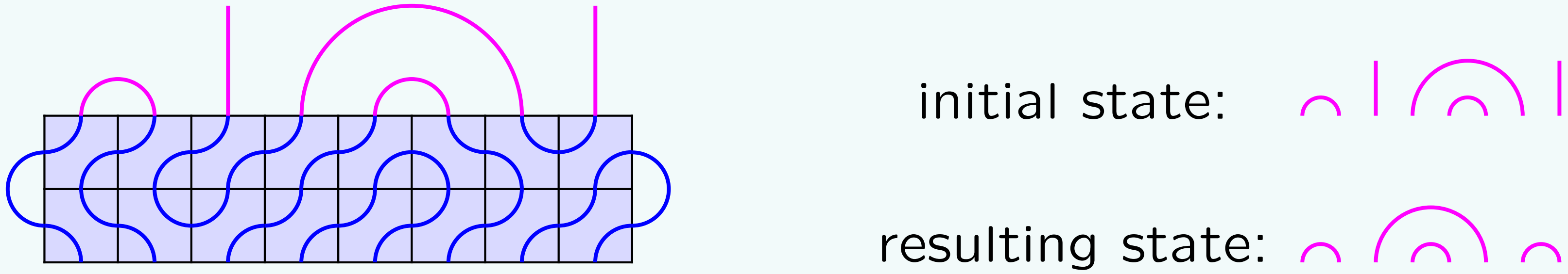


Defects

- Number of defects: ℓ
- For $N = 4$ and $\ell = 2$, there is a basis of 3 link states:

- By construction: $N - \ell \equiv 0 \pmod 2$

Example

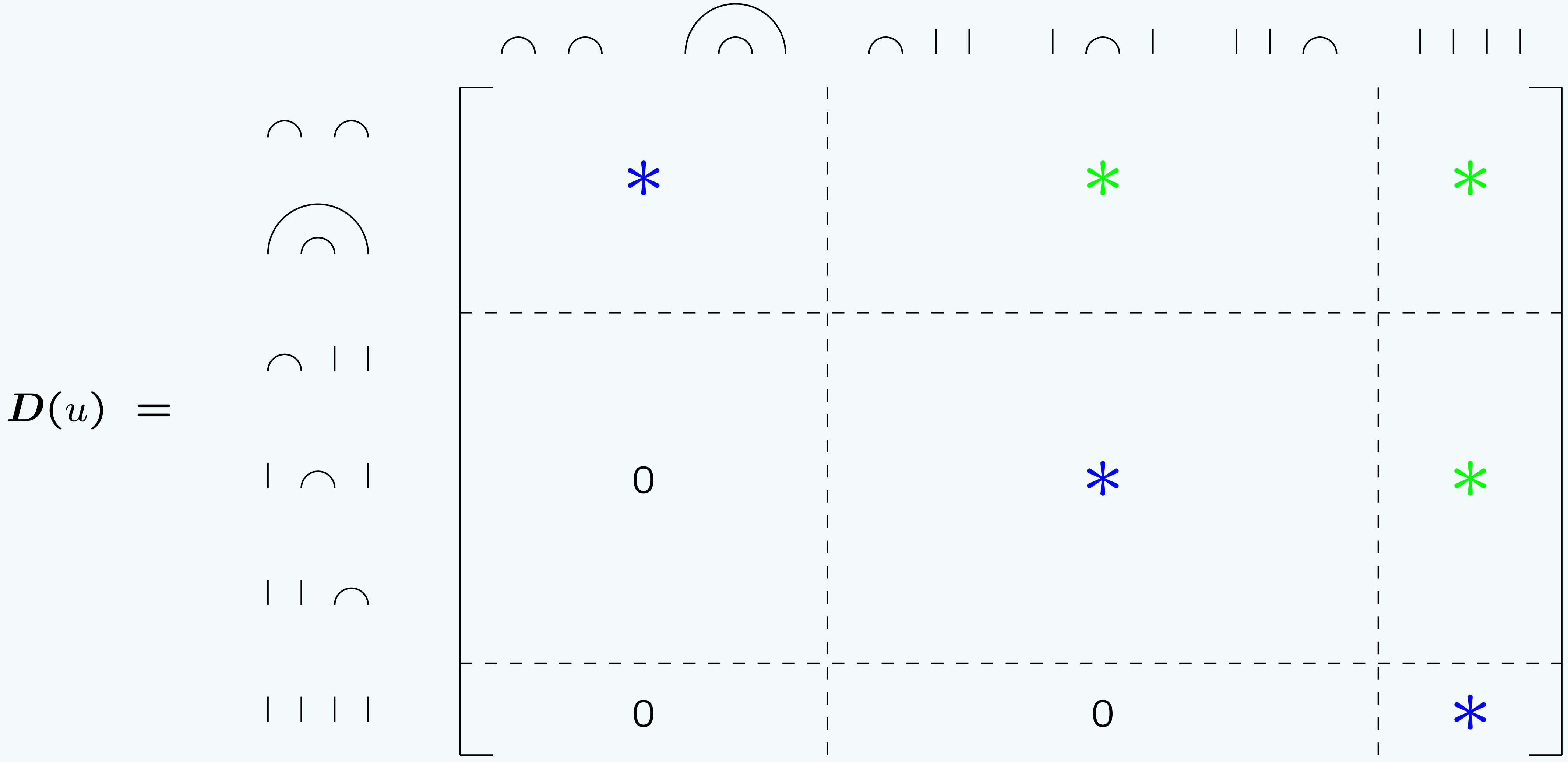

initial state:

resulting state:

- Defects can be annihilated in pairs but not created under the action of TL.

Upper Block-Triangular Matrix Representation

Example ($N = 4$)



$$D(u) = \left[\begin{array}{c:c:c} * & * & * \\ \hdashline 0 & * & * \\ \hdashline 0 & 0 & * \end{array}\right]$$

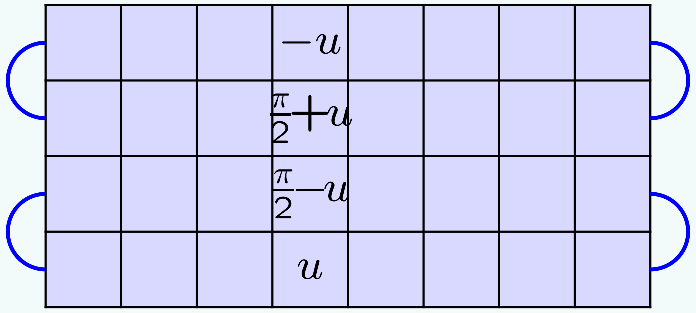
- Individually, the blocks on the **diagonal** can be diagonalized.
- In general, $D(u)$ is non-diagonalizable.

Inversion Identity

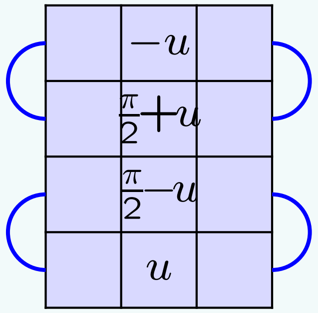
$$\boxed{\boldsymbol{D}(u)\boldsymbol{D}(u+\tfrac{\pi}{2}) \;=\; \left(\frac{\cos^{2N}u-\sin^{2N}u}{\cos^2 u-\sin^2 u}\right)^2 \boldsymbol{I}}$$

Proof

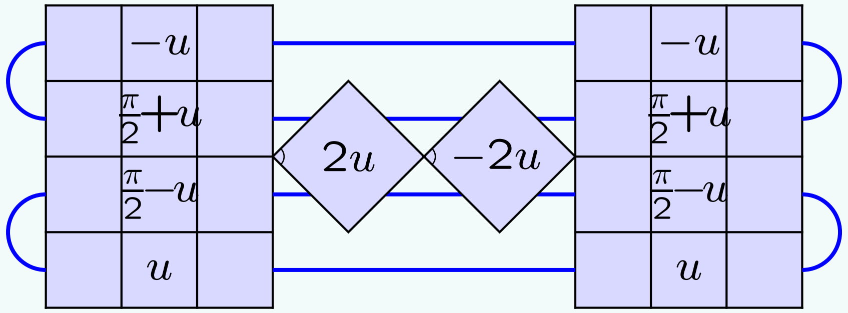
$$\boldsymbol{D}(u)\boldsymbol{D}(u+\frac{\pi}{2}) \;=\; -\frac{1}{\sin^2 2u}$$



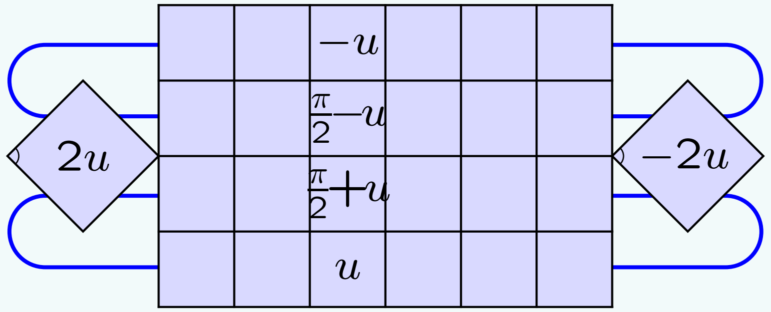
where

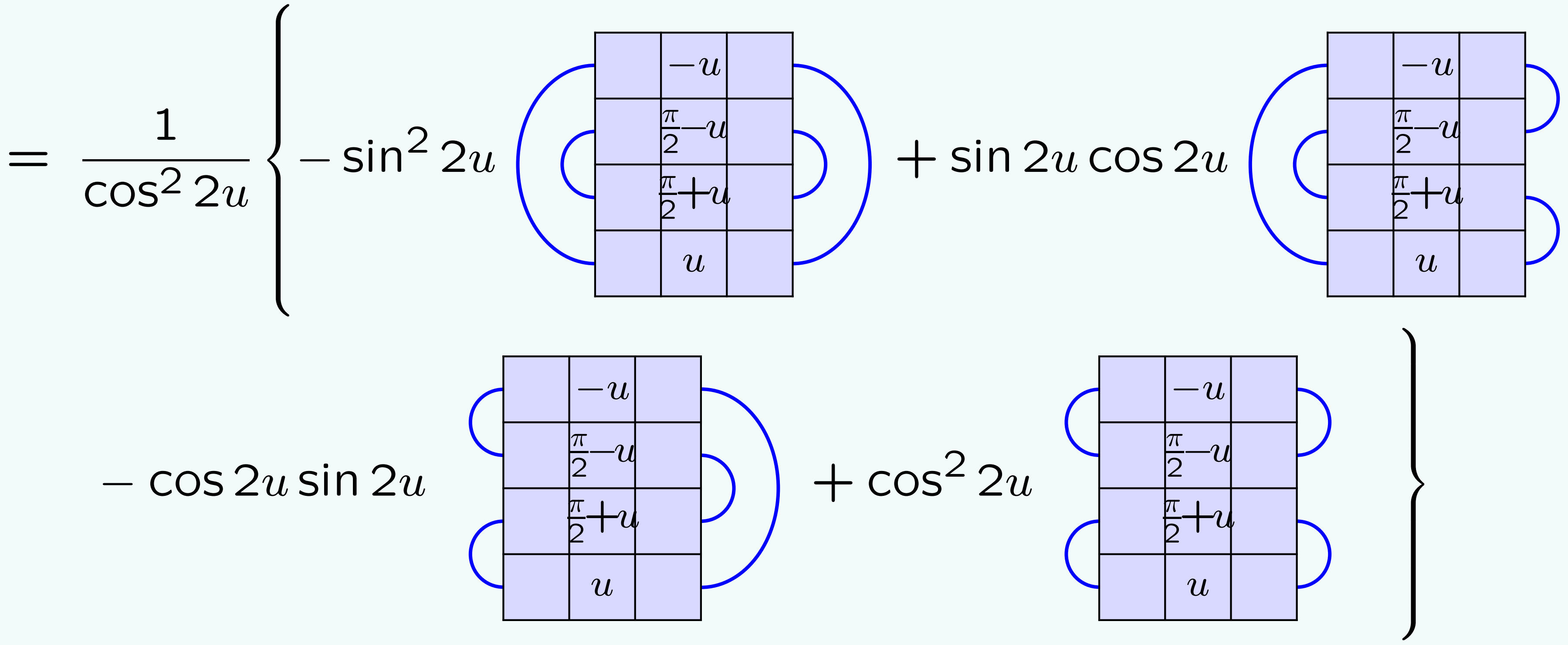


$$= \frac{1}{\cos^2 2u}$$

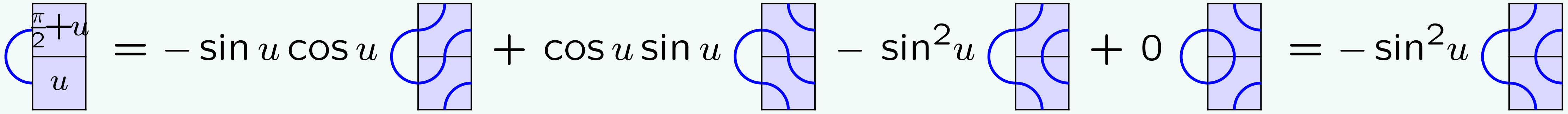


$$= \frac{1}{\cos^2 2u}$$

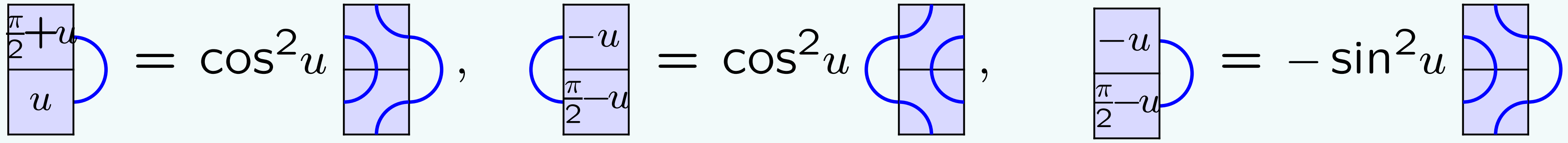




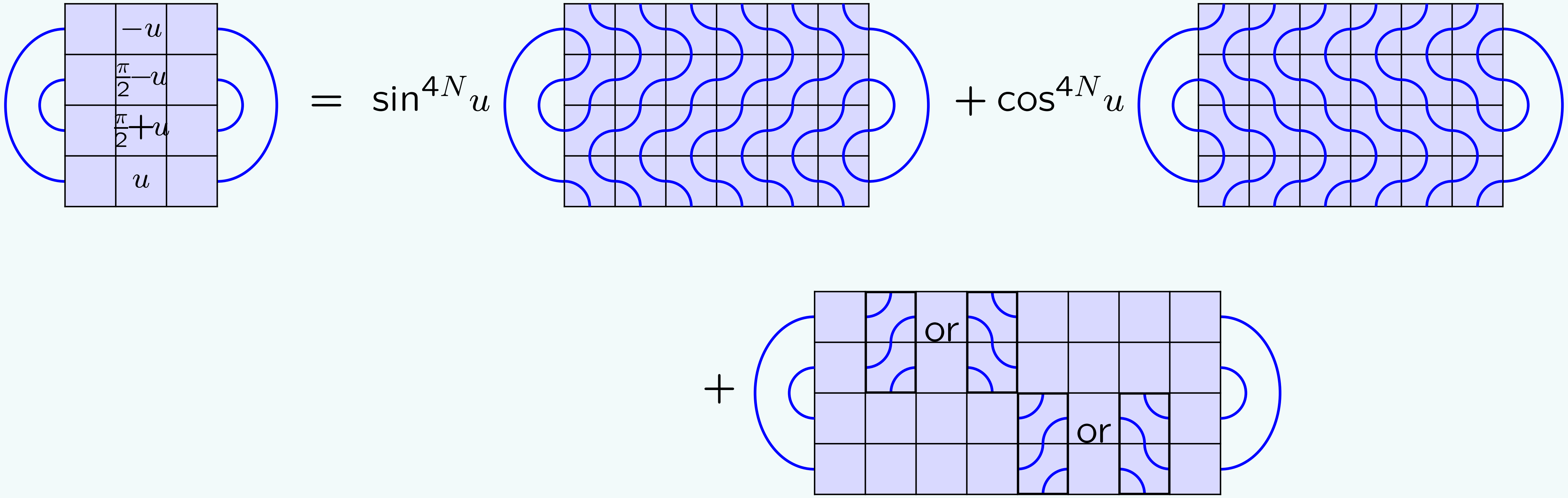
Certain half-arcs propagate since



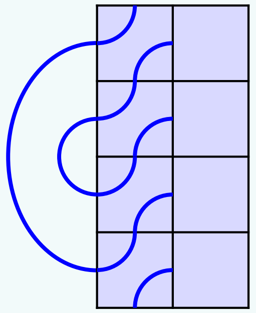
and similarly



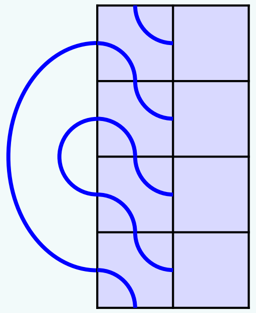
We have



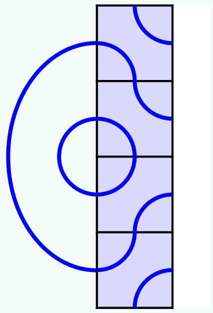
Reading from the left



$=$

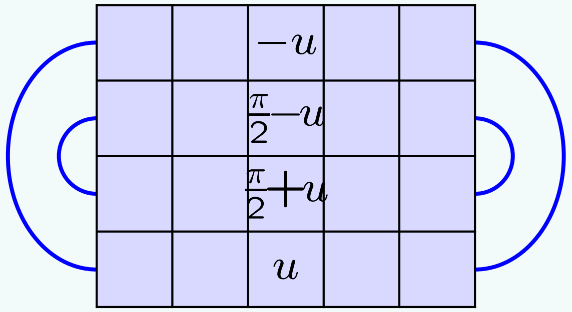


$= 0\,,$

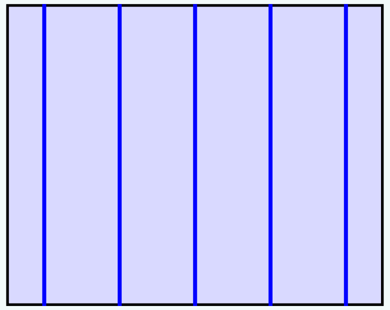


$= 0$

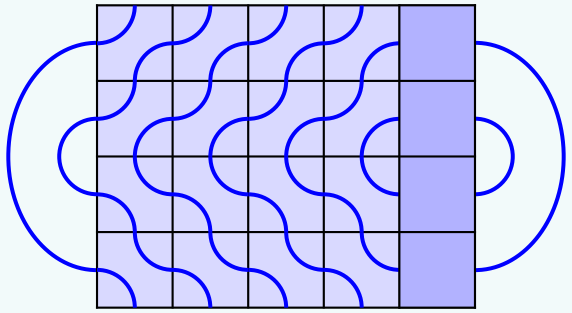
This implies



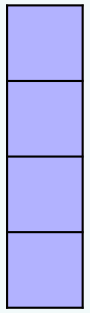
$= (\cos^{4N}u + \sin^{4N}u)$



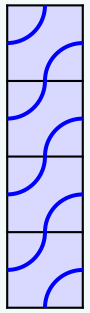
$+ \ (\cos u \sin u)^{2(N-1)}$



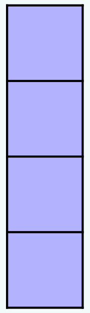
where



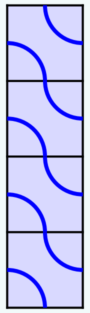
$= -\cos^2 u \sin^2 u$



or



$= -\cos^2 u \sin^2 u$



□

Eigenvalues

Functional Relation

$$\boxed{D(u)D(u+\frac{\pi}{2}) \;=\; \left(\frac{\cos^{2N}u-\sin^{2N}u}{\cos^2 u-\sin^2 u}\right)^2}$$

subject to

$$D(0) \;=\; 1, \qquad D(\frac{\pi}{2}-u) \;=\; D(u)$$

General Solution

$$\boxed{D(u) \;=\; \begin{cases} \dfrac{N}{2^{N-1}} \displaystyle\prod_{j=1}^{\frac{N}{2}-1} \Big(\frac{1}{\sin\frac{j\pi}{N}}+\epsilon_j \sin 2u\Big)\Big(\frac{1}{\sin\frac{j\pi}{N}}+\mu_j \sin 2u\Big), & N \text{ even} \\ \dfrac{1}{2^{N-1}} \displaystyle\prod_{j=1}^{\frac{N-1}{2}} \Big(\frac{1}{\sin\frac{(2j-1)\pi}{2N}}+\epsilon_j \sin 2u\Big)\Big(\frac{1}{\sin\frac{(2j-1)\pi}{2N}}+\mu_j \sin 2u\Big), & N \text{ odd} \end{cases}}$$

where $\epsilon_j^2=\mu_j^2=1$ for all j.

- Number of link states with ℓ defects: $\binom{N}{\frac{N-\ell}{2}}-\binom{N}{\frac{N-\ell-2}{2}}$ → **Selection Rules**
- Ground state: $\epsilon_j=\mu_j=1$ for all j.
- Excitations: some $\epsilon_j, \mu_j=-1$.

Finite-Size Corrections

- The partition function for a $2N' \times N$ strip reads

$$Z_{N,N'} \;=\; \mathrm{Tr}\,\boldsymbol{D}(u)^{N'} \;=\; \sum_n D_n(u)^{N'} \;=\; \sum_n e^{-N'E_n(u)}$$

where an eventual conformal invariance dictates that

$$\boxed{E_n(u) \;=\; -\ln D_n(u) \;\simeq\; 2Nf_{bulk}+f_{bdy}+\frac{2\pi\sin 2u}{N}\Big(-\frac{c}{24}+\Delta+k\Big)}$$

- The bulk and boundary free energies are

$$\boxed{f_{bulk} \;=\; \ln\sqrt{2}-\frac{1}{\pi}\int_0^{\pi/2}\ln\Big(\frac{1}{\sin t}+\sin 2u\Big)dt, \qquad f_{bdy} \;=\; \ln(1+\sin 2u)}$$

- The conformal corrections also follow from the Euler-Maclaurin formula and one finds

$$\boxed{c=-2, \qquad \Delta \;=\; \Delta_{1,s} \;=\; \frac{(2-s)^2-1}{8}, \quad s=1,2,3,\ldots}$$

where $s=\ell+1$. This corresponds to the first column in the extended Kac table.

- The excitations can be organized in finitized characters

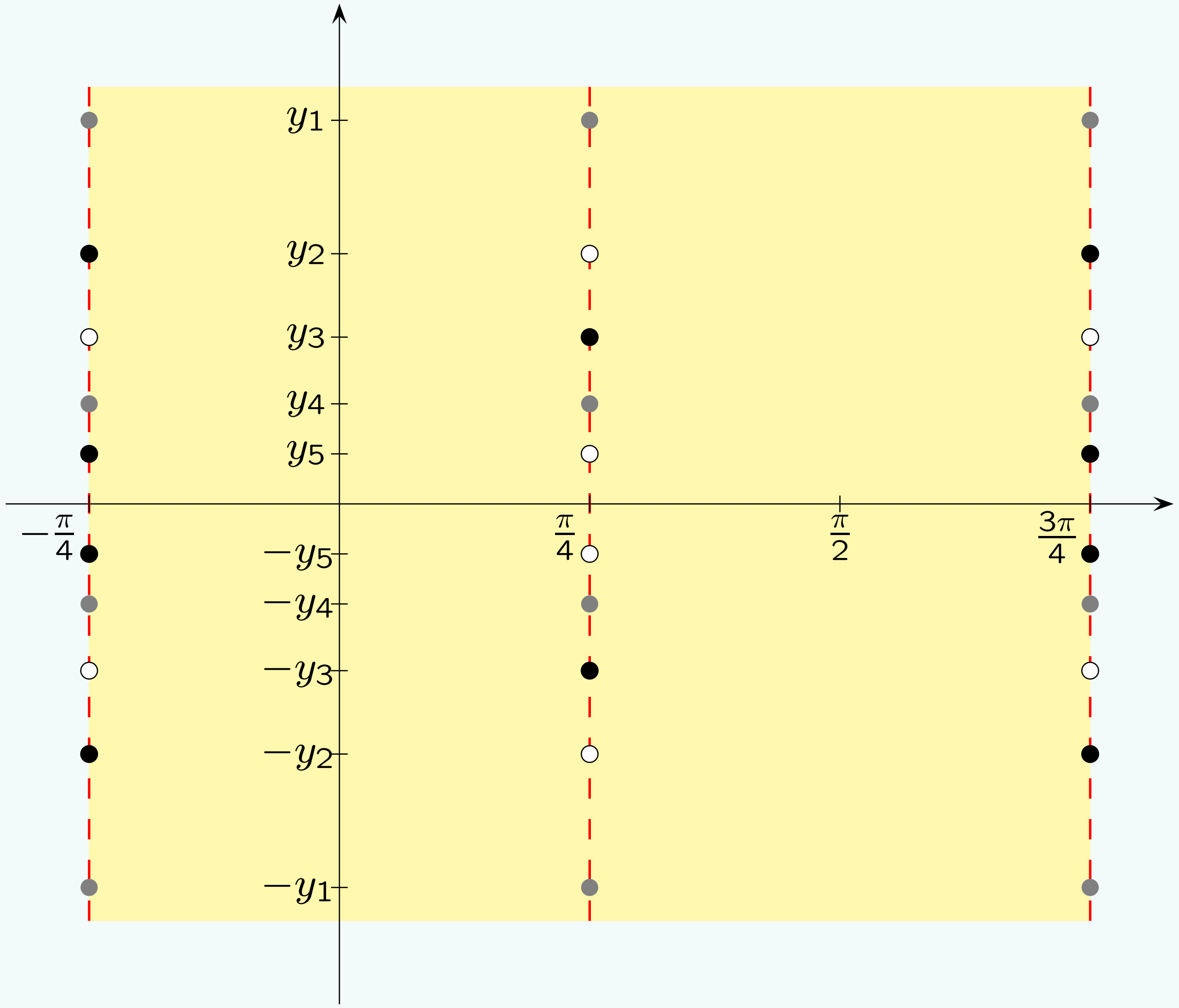
$$\boxed{\chi_{1,s}^{(N)}(q) \;=\; q^{-\frac{c}{24}+\Delta_{1,s}}\Big(\begin{bmatrix} N \\ \frac{N-s+1}{2}\end{bmatrix}_q - q^s\begin{bmatrix} N \\ \frac{N-s-1}{2}\end{bmatrix}_q\Big)}$$

where $q=\exp\Big(\frac{-2\pi N'\sin 2u}{N}\Big)$ while $\begin{bmatrix} n \\ m\end{bmatrix}_q$ is a q-binomial.

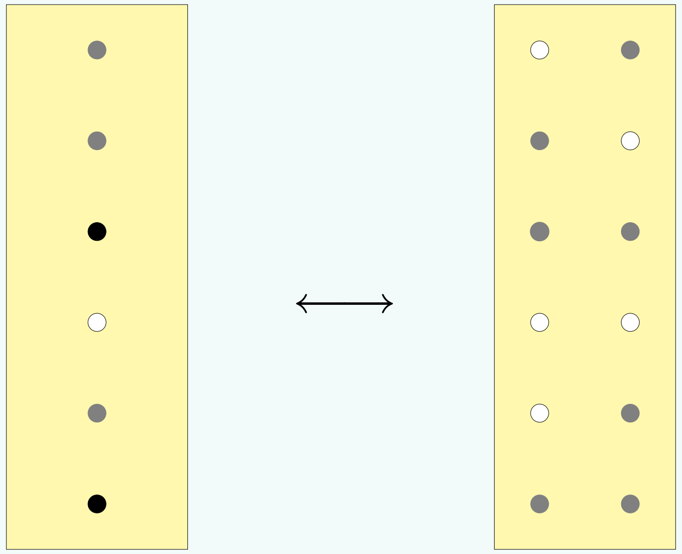
Selection Rules

- The selection rules can be described using **physical combinatorics** related to the patterns of zeros of $D(u)$ in the complex u-plane.
- We recall that $D(u) \sim \prod_j \left(\frac{1}{\sin t_j} + \epsilon_j \sin 2u\right)\left(\frac{1}{\sin t_j} + \mu_j \sin 2u\right)$ where $t_j = j\pi/N$ for N even, while $t_j = (2j-1)\pi/2N$ for N odd.
- With a period of π, the zeros are given by

$$\boxed{u \in \{(2+\nu_j)\frac{\pi}{4} \pm y_j\}, \quad \nu_j \in \{\epsilon_j, \mu_j\}, \quad y_j \;=\; -\frac{i}{2}\ln\tan\frac{t_j}{2}}$$



Double-Column Configurations



- Height: $M = 6$
- Signature: $L = (5, 4, 1)$, $R = (6, 4, 2, 1)$
- Occupancy: $m = |L| = 3$, $n = |R| = 4$
- Partial ordering: $S \preceq S'$ if $S_j \leq S'_j$, $j = 1, \ldots, |S|$

Admissible

$$\boxed{(L, R) \textit{ is admissible if } L \preceq R}$$

– in particular, $0 \leq m \leq n \leq M$

- Geometrically: signature links have non-negative slopes.
- Weight:

$$w(L, R) = \sum_{j=1}^{m} L_j + \sum_{j=1}^{n} R_j = 23$$

Associated Monomial

$$q^w = q^{23}$$

Generalized q-Narayana Numbers

Sum of Admissible Configurations

- For given M, m and n, the sum of the monomials associated to the admissible double-column configurations is given by

$$\boxed{\begin{aligned} \left\langle {M \atop m,n} \right\rangle_q &= q^{\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)} \left\{ {M \atop m,n} \right\}_q \\ &= q^{\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)} \Big(\begin{bmatrix} M \\ m \end{bmatrix}_q \begin{bmatrix} M \\ n \end{bmatrix}_q - q^{n-m+1} \begin{bmatrix} M \\ n+1 \end{bmatrix}_q \begin{bmatrix} M \\ m-1 \end{bmatrix}_q \Big) \end{aligned}}$$

- Fermionic –all coefficients are non-negative.
- For $0 \le m \le n \le M$,

$$\boxed{\left\{ {M \atop m,n} \right\}_q = 1 + \mathcal{O}(q)}$$

q-Narayana Numbers

$$\boxed{N_q(M,m) = q^{m(m+1)} \begin{bmatrix} M \\ m \end{bmatrix}_q \begin{bmatrix} M \\ m+1 \end{bmatrix}_q \frac{1-q}{1-q^M} = \left\langle {M-1 \atop m,m} \right\rangle_q}$$

Characters

Finitized Characters from Selection Rules

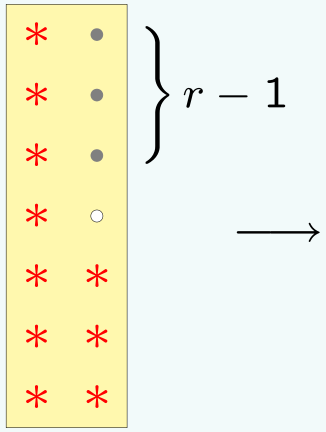
$$\chi_{1,s}^{(N)}(q) = \begin{cases} q^{\frac{1}{12}}\Bigg(\displaystyle\sum_{m=0}^{\frac{N-s+1}{2}} \left\langle {\frac{N-2}{2} \atop m,\, m+\frac{s-3}{2}} \right\rangle_q + \sum_{m=0}^{\frac{N-s-1}{2}} \left\langle {\frac{N-2}{2} \atop m,\, m+\frac{s-1}{2}} \right\rangle_q\Bigg), & s \text{ odd } (N \text{ even}) \\ q^{-\frac{1}{24}-\frac{s-2}{4}} \displaystyle\sum_{m=0}^{\frac{N-s+1}{2}} \left\langle {\frac{N-1}{2} \atop m,\, m+\frac{s-2}{2}} \right\rangle_q q^{-m}, & s \text{ even } (N \text{ odd}) \end{cases}$$

$$= q^{\Delta_{1,s}-\frac{c}{24}}\left(\begin{bmatrix} N \\ \frac{N-s+1}{2} \end{bmatrix}_q - q^s \begin{bmatrix} N \\ \frac{N-s-1}{2} \end{bmatrix}_q\right)$$

- In the continuum scaling limit, these finitized characters become the Virasoro characters

$$\boxed{\chi_{1,s}(q) = \lim_{N\to\infty} \chi_{1,s}^{(N)}(q) = q^{\frac{1}{12}} \frac{q^{\Delta_{1,s}} - q^{\Delta_{1,-s}}}{\prod_{n=1}^{\infty}(1-q^n)} = q^{\frac{1}{12}} \frac{q^{\Delta_{1,s}}(1-q^s)}{\prod_{n=1}^{\infty}(1-q^n)}}$$

Generalization

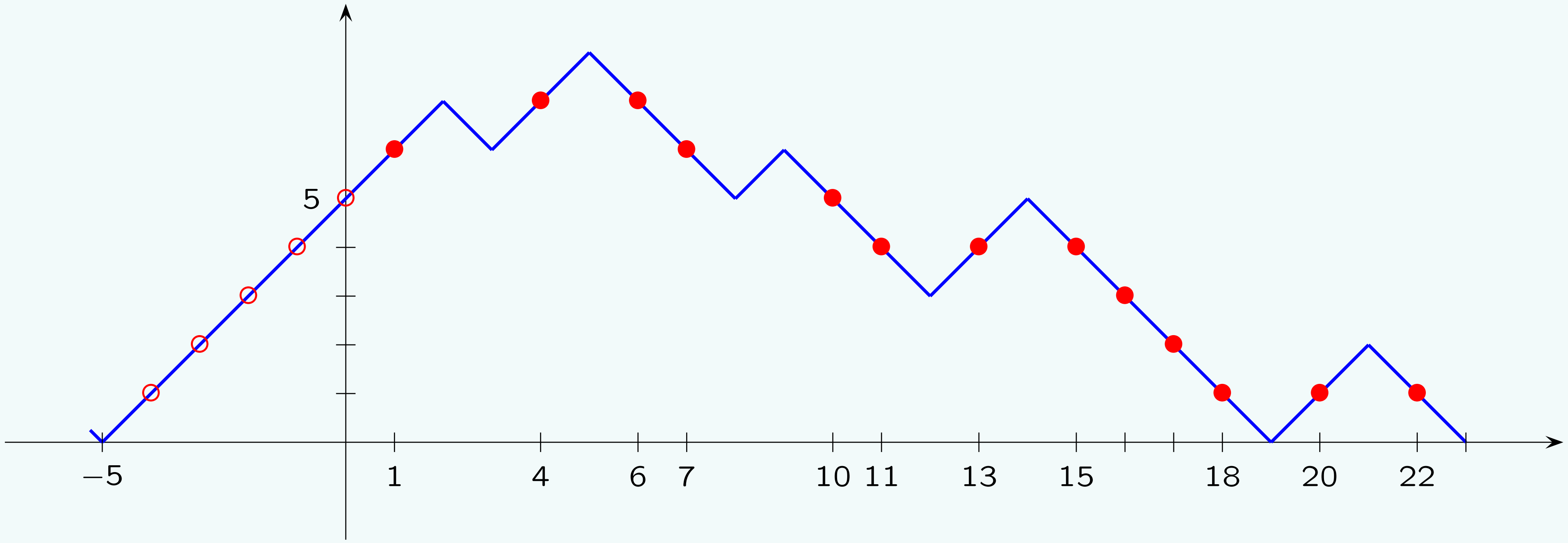


$$\longrightarrow \left\langle {M \atop m, n; r} \right\rangle_q, \qquad \chi_{r,s}^{(N)}(q) = [\text{Fermionic expression as above}]$$

$$= q^{\Delta_{r,s}-\frac{c}{24}}\left(\begin{bmatrix} N \\ \frac{N-s+r}{2} \end{bmatrix}_q - q^{rs} \begin{bmatrix} N \\ \frac{N-s-r}{2} \end{bmatrix}_q\right)$$

Relation to Dyck Paths

Example $(s = 6,\ \ N = 23)$



up ⟷ left ●

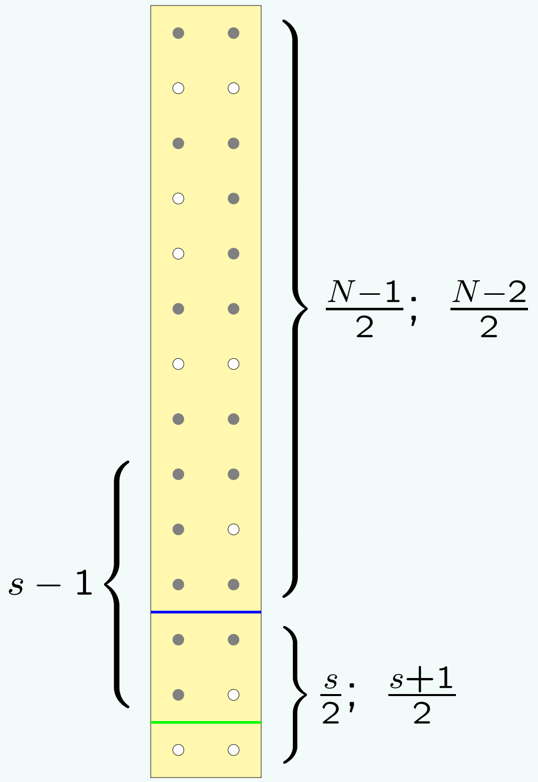
valley ⟷ left ○

down ⟷ right ●

peak ⟷ right ○

bijection

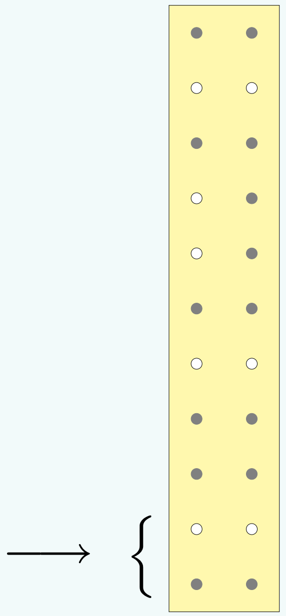
$M = \frac{N+s-1}{2},\ \ n = m$



$M = \frac{N-1}{2};\ \ \frac{N-2}{2},\ \ n = m + \frac{s-2}{2};\ \ m + \frac{s-3}{2},\ \ m + \frac{s-1}{2}$

$\binom{\bullet}{\bullet} - \binom{\bullet}{\circ} = \binom{\circ}{\bullet}$

bijection



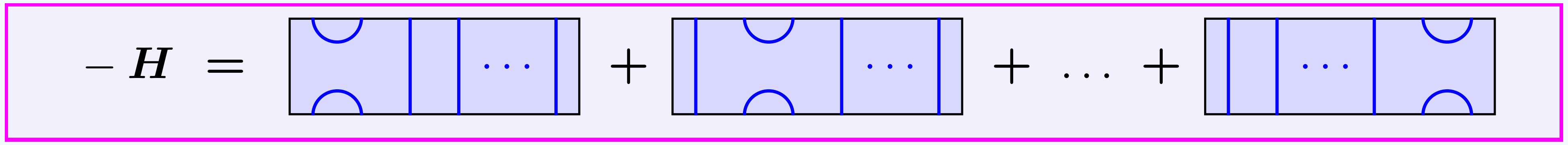
-extends to (r, s)

Hamiltonian Limits

Expansion of $D(u)$

$$\boldsymbol{D}(u) = \boldsymbol{I} - 2u\boldsymbol{H} + \mathcal{O}(u^2)$$

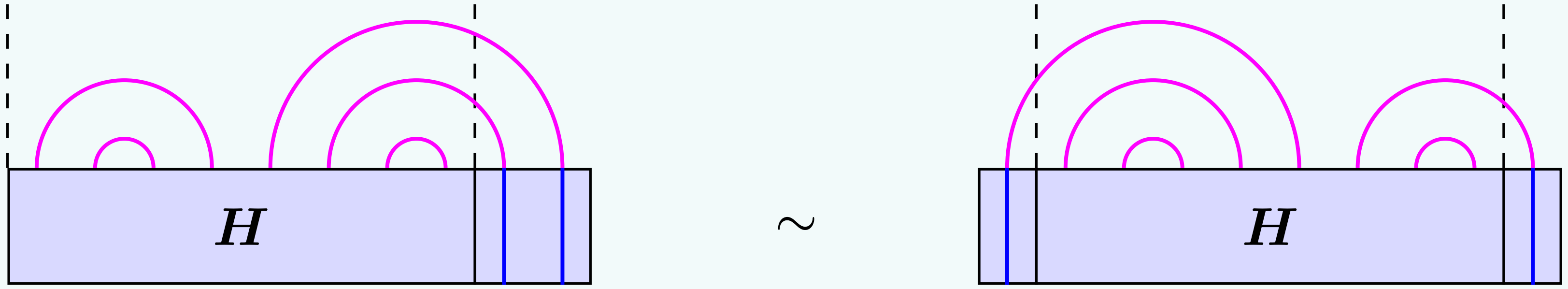
It follows that



- In terms of the generators of the *linear* TL algebra, this corresponds to

$$\boldsymbol{H} = -\sum_{j=1}^{N-1} e_j$$

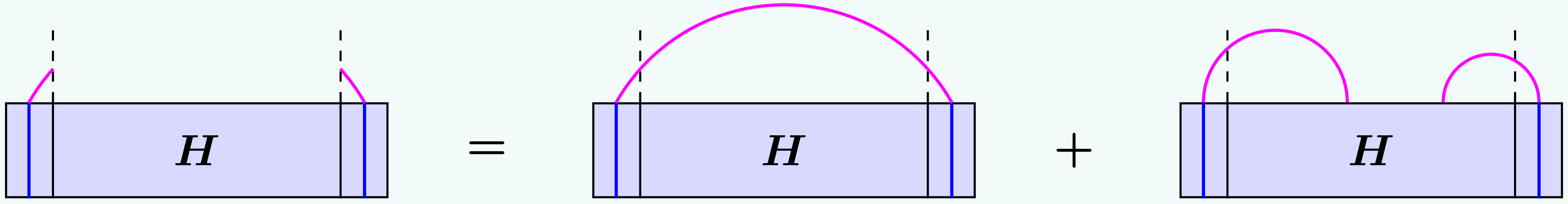
Defects Revisited



Fusion

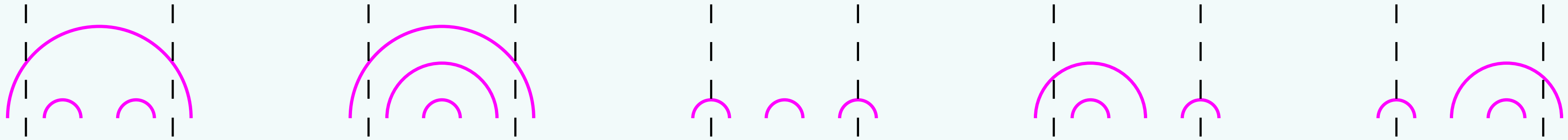
-Diagrammatic Implementation-

Example: $(1,2) \otimes_f (1,2)$



$(1,2) \otimes_f (1,2)$ $(1,1)$ $(1,3)$

- For $N = 4$, there are 5 link states:



Jordan Decomposition

$$\mathcal{H}_{(1,2)|(1,2)} = -\begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \sqrt{2}I \longrightarrow \left(\begin{array}{cc|c|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cline{1-3} 0 & 0 & \sqrt{2} & 0 & 0 \\ \cline{3-5} 0 & 0 & 0 & \sqrt{8} & 1 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{array}\right)$$

- Jordan cells are present for the **smallest** system sizes which can accommodate them.
- If a Jordan cell corresponding to a particular energy level is present for a particular system size, a Jordan cell corresponding to the **same** energy level is present for **all** larger system sizes.

Finitized Dilatation Generator

$$\boxed{\mathcal{H}_{(1,2)|(1,2)} \;\longrightarrow\; L_0^{(1,2)|(1,2)} \;=\; \mathrm{diag}\left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, 1, \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\right]}$$

Finitized Partition Function

$$Z^{(4)}_{(1,2)|(1,2)}(q) \;=\; \chi^{(4)}_{(1,1)}(q) + \chi^{(4)}_{(1,3)}(q) \;=\; q^{1/12}[(1+q^2)+(1+q+q^2)] \;=\; q^{1/12}(2+q+2q^2)$$

Fusion Rules

$$\boxed{(1,2) \otimes_f (1,2) \;=\; (1,1) \oplus_i (1,3) \;=\; \mathcal{R}_{1,1}}$$

and likewise

$$\begin{aligned}
(1,2) \otimes_f (1,3) \;&=\; (1,2) \oplus (1,4)\\
(1,2) \otimes_f \mathcal{R}_{1,1} \;&=\; (1,2) \oplus (1,2) \oplus (1,4)\\
\mathcal{R}_{1,1} \otimes_f \mathcal{R}_{1,1} \;&=\; \mathcal{R}_{1,1} \oplus \mathcal{R}_{1,1} \oplus \mathcal{R}_{2,1}\\
&\;\;\vdots
\end{aligned}$$

⟶ **Fusion Algebra**

Concluding Remarks

Summary

- Exactly solvable lattice model for critical dense polymers
- Inversion identity
- Finite-size corrections
- CFT with spectrum $c = -2$, $\Delta = \Delta_{1,s}$, $s = 1, 2, 3, \ldots$
- Physical combinatorics
- Link to Dyck paths
- Hamiltonian limits
- Diagrammatic implementation of fusion
- Jordan cells

Outlook

- Other geometries: cylinder and torus
- Other models: critical percolation, logarithmic Ising model, . . .