

Solvable Critical Dense Polymers

by

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based on

- P.A. Pearce, J. Rasmussen, J.-B. Zuber, *Logarithmic Minimal Models*, J. Stat. Mech. 0611 (2006) P017
- P.A. Pearce, J. Rasmussen, *Solvable Critical Dense Polymers*, hep-th/0610273
- P.A. Pearce, J. Rasmussen, *Physical Combinatorics of Critical Dense Polymers* (2007)

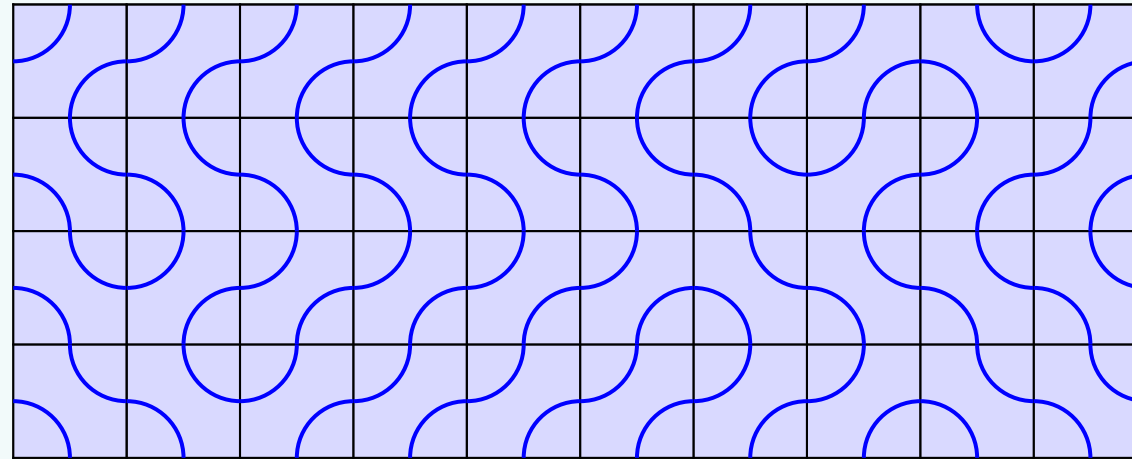
Outline

- Lattice model
- Planar Temperley-Lieb algebra
- Double-row transfer matrix and link states
- Inversion identity
- Finite-size corrections
- Physical combinatorics: selection rules and characters
- Relation to Dyck paths
- Hamiltonian limits
- Fusion and Jordan cells

Lattice Model and Planar TL Algebra

Typical Configuration

–dense loop representation–



–with some boundary conditions

Face Operators

$$\begin{array}{|c|} \hline u \\ \hline \end{array} = \sin u \begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} + \cos u \begin{array}{|c|} \hline \text{arc} \\ \hline \end{array}$$

- Spectral parameter: u
- Crossing parameter: $\lambda = \frac{\pi}{2}$
- Fugacity: $\beta = 2 \cos \lambda = 0 \longrightarrow$ **NO LOOPS**

Local Properties

Inversion Relation

$$\begin{aligned}
 & \text{Diagram 1} = \cos v \cos(-v) \text{Diagram 2} + \beta \sin v \sin(-v) \text{Diagram 3} \\
 & + \cos v \sin(-v) \text{Diagram 4} + \sin v \cos(-v) \text{Diagram 5} \\
 & = \cos^2 v \text{Diagram 6}
 \end{aligned}$$

The diagrams consist of light blue diamonds and squares with blue arcs and lines. The first diagram shows two diamonds with arcs between them, labeled v and $-v$. The second diagram shows two diamonds with arcs inside each. The third diagram shows two diamonds with arcs inside each and a central circle. The fourth diagram shows two diamonds with arcs inside each and a central circle. The fifth diagram shows two diamonds with arcs inside each and a central circle. The sixth diagram shows a single diamond with arcs inside.

Yang-Baxter Equation

$$\text{Diagram 1} = \text{Diagram 2}$$

The diagrams consist of light blue diamonds and squares with blue lines. The first diagram shows a diamond with a square attached to its right side, labeled $u-v$ and v , u . The second diagram shows a square with a diamond attached to its right side, labeled u , v and $u-v$.

Double-Row Transfer Matrix

Definition as N -tangle

$$D(u) = \frac{1}{\sin 2u} \begin{array}{|c|c|c|c|c|c|c|c|} \hline \frac{\pi}{2}-u & & \dots & & & \dots & & \frac{\pi}{2}-u \\ \hline u & & \dots & & & \dots & & u \\ \hline \end{array}$$

Normalization

$$\lim_{u \rightarrow 0} D(u) = I =$$

Commuting Family

$$D(u)D(v) = D(v)D(u)$$

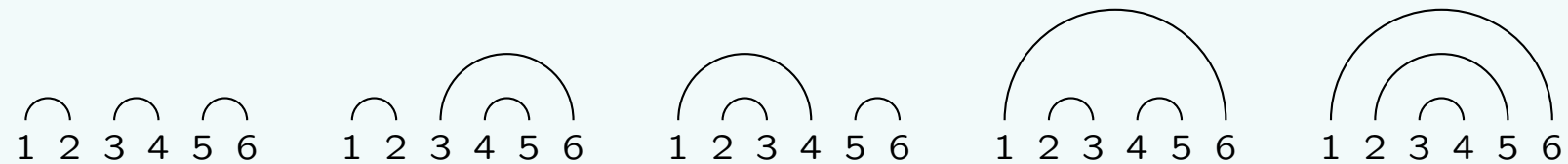
- Multiplication is vertical concatenation of diagrams
- Equality is the equality of N -tangles

Crossing Symmetry

$$D(u) = D(\lambda - u)$$

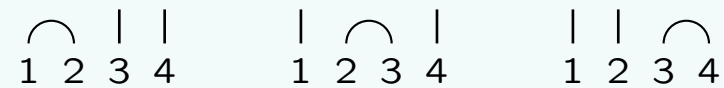
Link States

- In a 'fixed direction', the planar N -tangles can act on a vector space of link diagrams.
- Number of nodes: N
- For $N = 6$ and no defects, there is a basis of 5 link states:



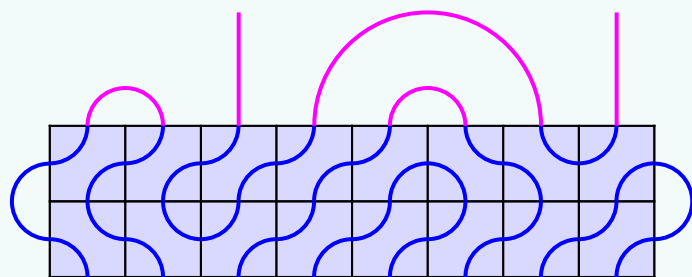
Defects

- Number of defects: ℓ
- For $N = 4$ and $\ell = 2$, there is a basis of 3 link states:



- By construction: $N - \ell \equiv 0 \pmod{2}$

Example



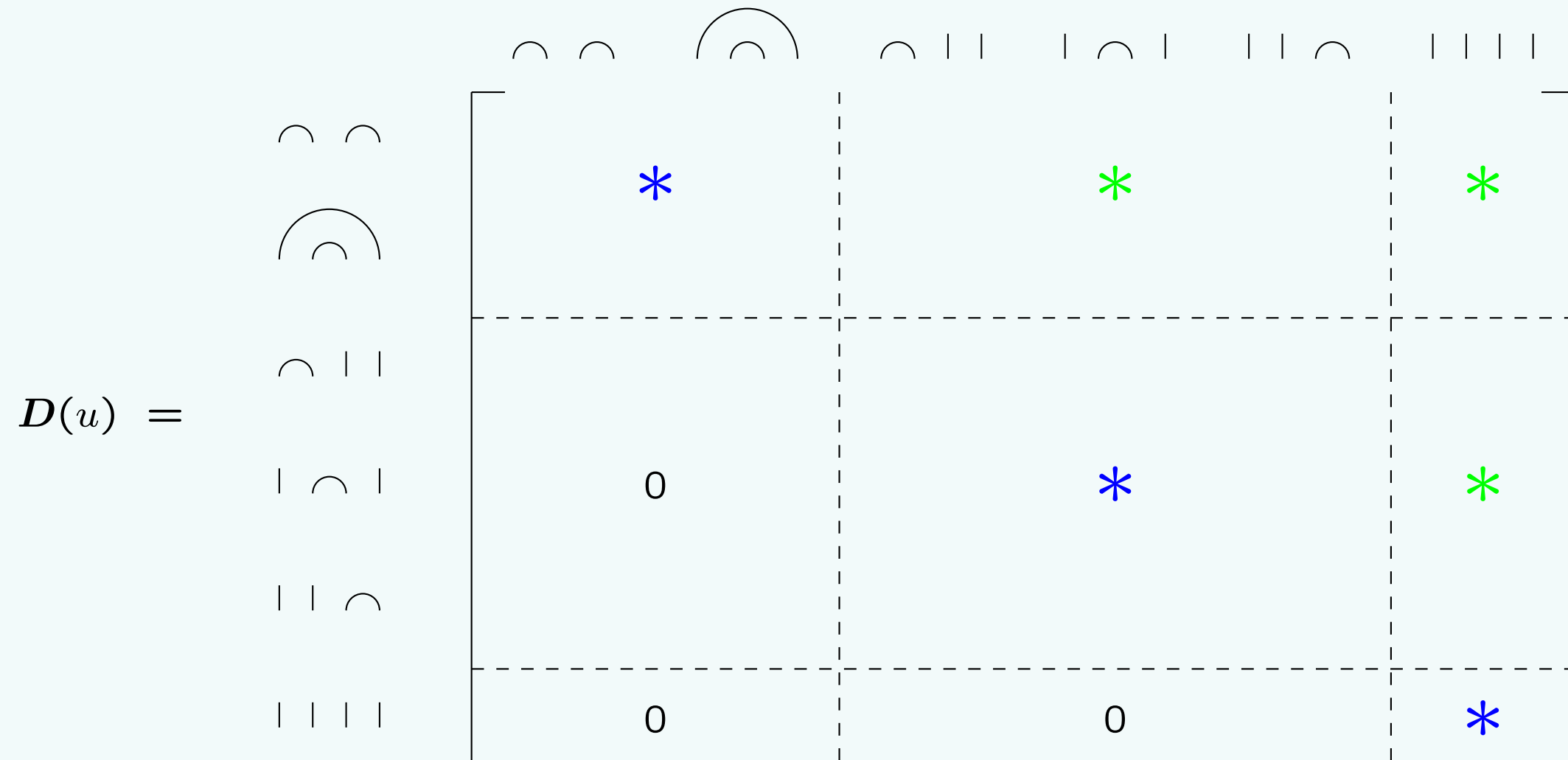
initial state:

resulting state:

- Defects can be annihilated in pairs but not created under the action of TL.

Upper Block-Triangular Matrix Representation

Example ($N = 4$)

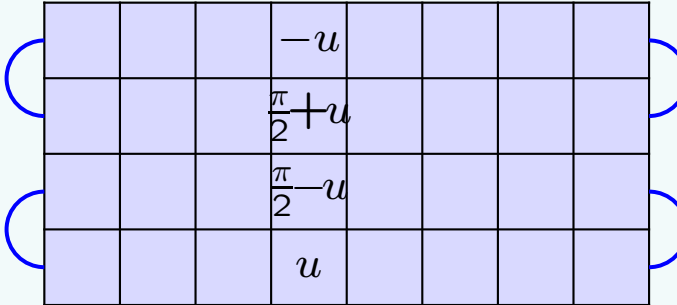


- Individually, the blocks on the **diagonal** can be diagonalized.
- In general, $D(u)$ is non-diagonalizable.

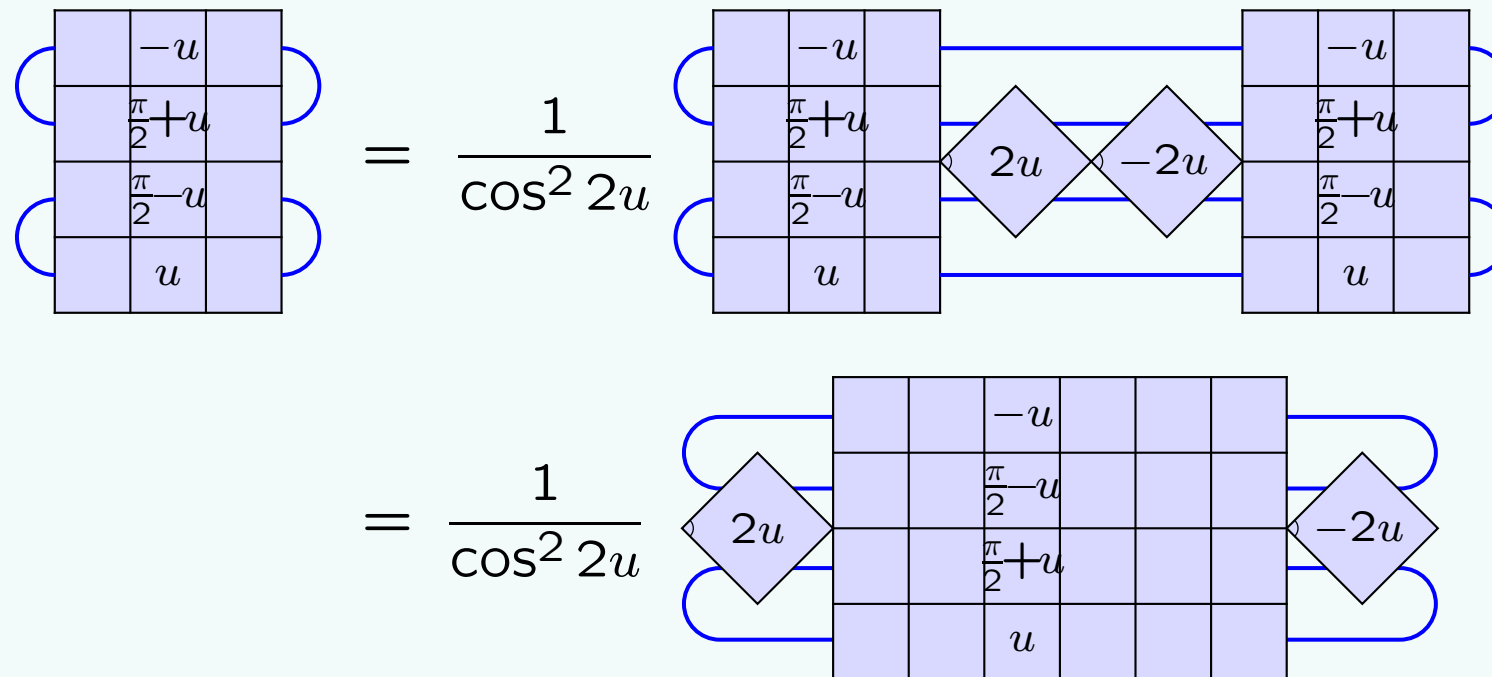
Inversion Identity

$$D(u)D(u + \frac{\pi}{2}) = \left(\frac{\cos^{2N} u - \sin^{2N} u}{\cos^2 u - \sin^2 u} \right)^2 \mathbf{I}$$

Proof

$$D(u)D(u + \frac{\pi}{2}) = -\frac{1}{\sin^2 2u}$$


where



$$= \frac{1}{\cos^2 2u}$$

$$= \frac{1}{\cos^2 2u}$$

$$\begin{aligned}
&= \frac{1}{\cos^2 2u} \left\{ -\sin^2 2u \left(\begin{array}{|c|c|c|} \hline & -u & \\ \hline & \frac{\pi}{2}-u & \\ \hline & \frac{\pi}{2}+u & \\ \hline & u & \\ \hline \end{array} \right) + \sin 2u \cos 2u \left(\begin{array}{|c|c|c|} \hline & -u & \\ \hline & \frac{\pi}{2}-u & \\ \hline & \frac{\pi}{2}+u & \\ \hline & u & \\ \hline \end{array} \right) \right. \\
&\quad \left. - \cos 2u \sin 2u \left(\begin{array}{|c|c|c|} \hline & -u & \\ \hline & \frac{\pi}{2}-u & \\ \hline & \frac{\pi}{2}+u & \\ \hline & u & \\ \hline \end{array} \right) + \cos^2 2u \left(\begin{array}{|c|c|c|} \hline & -u & \\ \hline & \frac{\pi}{2}-u & \\ \hline & \frac{\pi}{2}+u & \\ \hline & u & \\ \hline \end{array} \right) \right\}
\end{aligned}$$

Certain half-arcs propagate since

$$\begin{array}{|c|} \hline \frac{\pi}{2}+u \\ \hline u \\ \hline \end{array} = -\sin u \cos u \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right) + \cos u \sin u \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right) - \sin^2 u \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right) + 0 \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right) = -\sin^2 u \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right)$$

and similarly

$$\begin{array}{|c|} \hline \frac{\pi}{2}+u \\ \hline u \\ \hline \end{array} = \cos^2 u \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right), \quad \begin{array}{|c|} \hline -u \\ \hline \frac{\pi}{2}-u \\ \hline \end{array} = \cos^2 u \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right), \quad \begin{array}{|c|} \hline -u \\ \hline \frac{\pi}{2}-u \\ \hline \end{array} = -\sin^2 u \left(\begin{array}{|c|} \hline \text{arc} \\ \hline \end{array} \right)$$

We have

$$\begin{array}{|c|c|c|} \hline & -u & \\ \hline & \frac{\pi}{2}-u & \\ \hline & \frac{\pi}{2}+u & \\ \hline & u & \\ \hline \end{array}
 \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}
 = \sin^{4N} u
 \begin{array}{|c|c|c|c|c|c|c|c|} \hline \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} \\ \hline \end{array}
 + \cos^{4N} u
 \begin{array}{|c|c|c|c|c|c|c|c|} \hline \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} & \text{wavy} \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|c|c|c|c|c|} \hline & \text{or} & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline \end{array}
 \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}
 + \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline \end{array}
 \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}$$

Reading from the left

Diagrammatic equations showing the reduction of a grid with wavy lines to zero. The first equation shows a grid with wavy lines on the left side, which is equal to a grid with wavy lines on the right side, which is equal to zero. The second equation shows a grid with wavy lines on the left side, which is equal to zero.

This implies

Diagrammatic equation for the trace of a grid with wavy lines. The left side shows a grid with wavy lines on the left and right sides, with labels $-u$, $\frac{\pi}{2}-u$, $\frac{\pi}{2}+u$, and u in the middle column. This is equal to $(\cos^{4N}u + \sin^{4N}u)$ times a grid with vertical lines. The right side shows a grid with wavy lines on the left and right sides, with a shaded vertical strip in the middle, multiplied by $(\cos u \sin u)^{2(N-1)}$.

where

Diagrammatic equations for the definition of the shaded strip. The first equation shows a shaded vertical strip equal to $-\cos^2u \sin^2u$ times a grid with wavy lines. The second equation shows a shaded vertical strip equal to $-\cos^2u \sin^2u$ times a grid with wavy lines.

□

Eigenvalues

Functional Relation

$$D(u)D(u + \frac{\pi}{2}) = \left(\frac{\cos^{2N}u - \sin^{2N}u}{\cos^2u - \sin^2u} \right)^2$$

subject to

$$D(0) = 1, \quad D(\frac{\pi}{2} - u) = D(u)$$

General Solution

$$D(u) = \begin{cases} \frac{N}{2^{N-1}} \prod_{j=1}^{\frac{N}{2}-1} \left(\frac{1}{\sin \frac{j\pi}{N}} + \epsilon_j \sin 2u \right) \left(\frac{1}{\sin \frac{j\pi}{N}} + \mu_j \sin 2u \right), & N \text{ even} \\ \frac{1}{2^{N-1}} \prod_{j=1}^{\frac{N-1}{2}} \left(\frac{1}{\sin \frac{(2j-1)\pi}{2N}} + \epsilon_j \sin 2u \right) \left(\frac{1}{\sin \frac{(2j-1)\pi}{2N}} + \mu_j \sin 2u \right), & N \text{ odd} \end{cases}$$

where $\epsilon_j^2 = \mu_j^2 = 1$ for all j .

- Number of link states with ℓ defects: $\binom{N}{\frac{N-\ell}{2}} - \binom{N}{\frac{N-\ell-2}{2}} \rightarrow$ **Selection Rules**
- Ground state: $\epsilon_j = \mu_j = 1$ for all j .
- Excitations: some $\epsilon_j, \mu_j = -1$.

Finite-Size Corrections

- The partition function for a $2N' \times N$ strip reads

$$Z_{N,N'} = \text{Tr} \mathbf{D}(u)^{N'} = \sum_n D_n(u)^{N'} = \sum_n e^{-N' E_n(u)}$$

where an eventual conformal invariance dictates that

$$E_n(u) = -\ln D_n(u) \simeq 2N f_{bulk} + f_{bdy} + \frac{2\pi \sin 2u}{N} \left(-\frac{c}{24} + \Delta + k \right)$$

- The bulk and boundary free energies are

$$f_{bulk} = \ln \sqrt{2} - \frac{1}{\pi} \int_0^{\pi/2} \ln \left(\frac{1}{\sin t} + \sin 2u \right) dt, \quad f_{bdy} = \ln(1 + \sin 2u)$$

- The conformal corrections also follow from the Euler-Maclaurin formula and one finds

$$c = -2, \quad \Delta = \Delta_{1,s} = \frac{(2-s)^2 - 1}{8}, \quad s = 1, 2, 3, \dots$$

where $s = \ell + 1$. This corresponds to the first column in the extended Kac table.

- The excitations can be organized in finitized characters

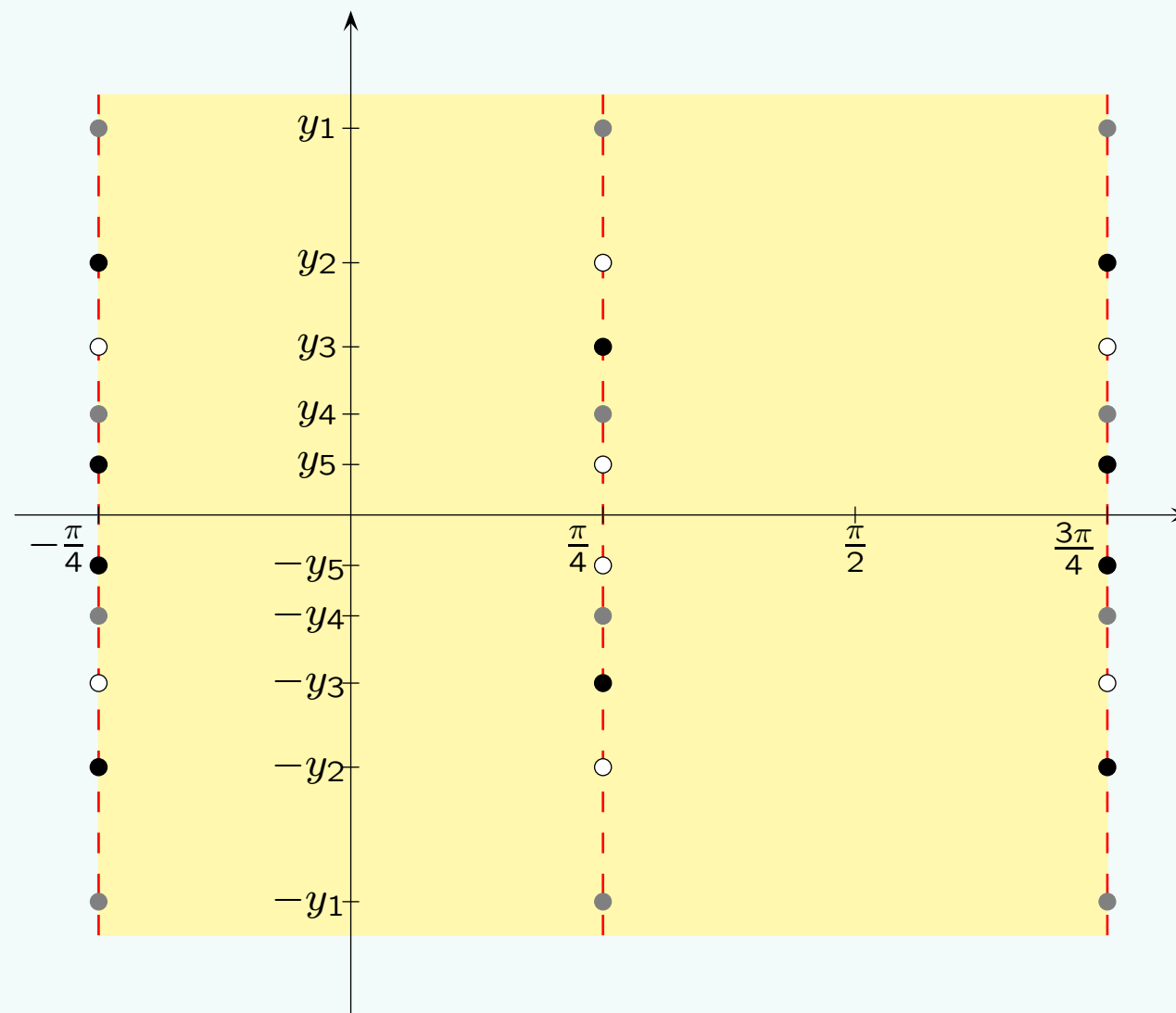
$$\chi_{1,s}^{(N)}(q) = q^{-\frac{c}{24} + \Delta_{1,s}} \left(\left[\frac{N}{\frac{N-s+1}{2}} \right]_q - q^s \left[\frac{N}{\frac{N-s-1}{2}} \right]_q \right)$$

where $q = \exp \left(\frac{-2\pi N' \sin 2u}{N} \right)$ while $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]_q$ is a q -binomial.

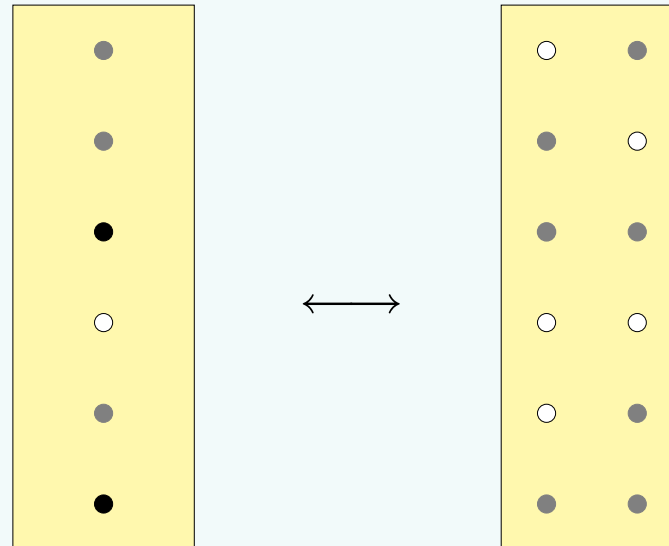
Selection Rules

- The selection rules can be described using **physical combinatorics** related to the patterns of zeros of $D(u)$ in the complex u -plane.
- We recall that $D(u) \sim \prod_j \left(\frac{1}{\sin t_j} + \epsilon_j \sin 2u \right) \left(\frac{1}{\sin t_j} + \mu_j \sin 2u \right)$ where $t_j = j\pi/N$ for N even, while $t_j = (2j - 1)\pi/2N$ for N odd.
- With a period of π , the zeros are given by

$$u \in \left\{ \left(2 + \nu_j \right) \frac{\pi}{4} \pm y_j \right\}, \quad \nu_j \in \{ \epsilon_j, \mu_j \}, \quad y_j = -\frac{i}{2} \ln \tan \frac{t_j}{2}$$



Double-Column Configurations



- Height: $M = 6$
- Signature: $L = (5, 4, 1)$, $R = (6, 4, 2, 1)$
- Occupancy: $m = |L| = 3$, $n = |R| = 4$
- Partial ordering: $S \preceq S'$ if $S_j \leq S'_j$, $j = 1, \dots, |S|$

Admissible

(L, R) is admissible if $L \preceq R$

– in particular, $0 \leq m \leq n \leq M$

- Geometrically: signature links have non-negative slopes.
- Weight:

$$w(L, R) = \sum_{j=1}^m L_j + \sum_{j=1}^n R_j = 23$$

Associated Monomial

$$q^w = q^{23}$$

Generalized q -Narayana Numbers

Sum of Admissible Configurations

- For given M , m and n , the sum of the monomials associated to the admissible double-column configurations is given by

$$\begin{aligned} \left\langle \begin{matrix} M \\ m, n \end{matrix} \right\rangle_q &= q^{\frac{1}{2}m(m+1) + \frac{1}{2}n(n+1)} \left\{ \begin{matrix} M \\ m, n \end{matrix} \right\}_q \\ &= q^{\frac{1}{2}m(m+1) + \frac{1}{2}n(n+1)} \left(\begin{bmatrix} M \\ m \end{bmatrix}_q \begin{bmatrix} M \\ n \end{bmatrix}_q - q^{n-m+1} \begin{bmatrix} M \\ n+1 \end{bmatrix}_q \begin{bmatrix} M \\ m-1 \end{bmatrix}_q \right) \end{aligned}$$

- Fermionic –all coefficients are non-negative.
- For $0 \leq m \leq n \leq M$,

$$\left\{ \begin{matrix} M \\ m, n \end{matrix} \right\}_q = 1 + \mathcal{O}(q)$$

q -Narayana Numbers

$$N_q(M, m) = q^{m(m+1)} \begin{bmatrix} M \\ m \end{bmatrix}_q \begin{bmatrix} M \\ m+1 \end{bmatrix}_q \frac{1-q}{1-q^M} = \left\langle \begin{matrix} M-1 \\ m, m \end{matrix} \right\rangle_q$$

Characters

Finitized Characters from Selection Rules

$$\chi_{1,s}^{(N)}(q) = \begin{cases} q^{\frac{1}{12}} \left(\sum_{m=0}^{\frac{N-s+1}{2}} \left\langle m, m + \frac{s-3}{2} \right\rangle_q + \sum_{m=0}^{\frac{N-s-1}{2}} \left\langle m, m + \frac{s-1}{2} \right\rangle_q \right), & s \text{ odd } (N \text{ even}) \\ q^{-\frac{1}{24} - \frac{s-2}{4}} \sum_{m=0}^{\frac{N-s+1}{2}} \left\langle m, m + \frac{s-2}{2} \right\rangle_q q^{-m}, & s \text{ even } (N \text{ odd}) \end{cases}$$

$$= q^{\Delta_{1,s} - \frac{c}{24}} \left(\left[\frac{N}{\frac{N-s+1}{2}} \right]_q - q^s \left[\frac{N}{\frac{N-s-1}{2}} \right]_q \right)$$

- In the continuum scaling limit, these finitized characters become the Virasoro characters

$$\chi_{1,s}(q) = \lim_{N \rightarrow \infty} \chi_{1,s}^{(N)}(q) = q^{\frac{1}{12}} \frac{q^{\Delta_{1,s}} - q^{\Delta_{1,-s}}}{\prod_{n=1}^{\infty} (1 - q^n)} = q^{\frac{1}{12}} \frac{q^{\Delta_{1,s}} (1 - q^s)}{\prod_{n=1}^{\infty} (1 - q^n)}$$

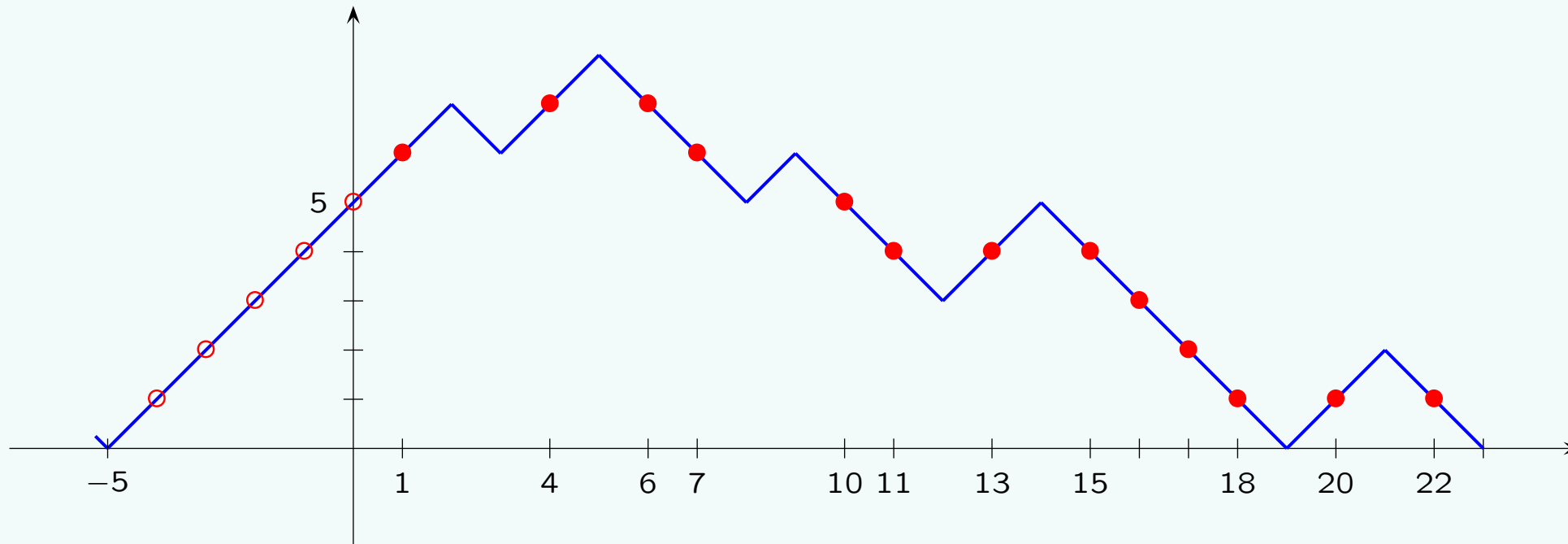
Generalization

$$\left. \begin{array}{l} * \bullet \\ * \bullet \\ * \bullet \\ * \circ \\ * * \\ * * \\ * * \end{array} \right\} r-1 \longrightarrow \left\langle \begin{array}{c} M \\ m, n; r \end{array} \right\rangle_q, \quad \chi_{r,s}^{(N)}(q) = [\text{Fermionic expression as above}]$$

$$= q^{\Delta_{r,s} - \frac{c}{24}} \left(\left[\frac{N}{\frac{N-s+r}{2}} \right]_q - q^{rs} \left[\frac{N}{\frac{N-s-r}{2}} \right]_q \right)$$

Relation to Dyck Paths

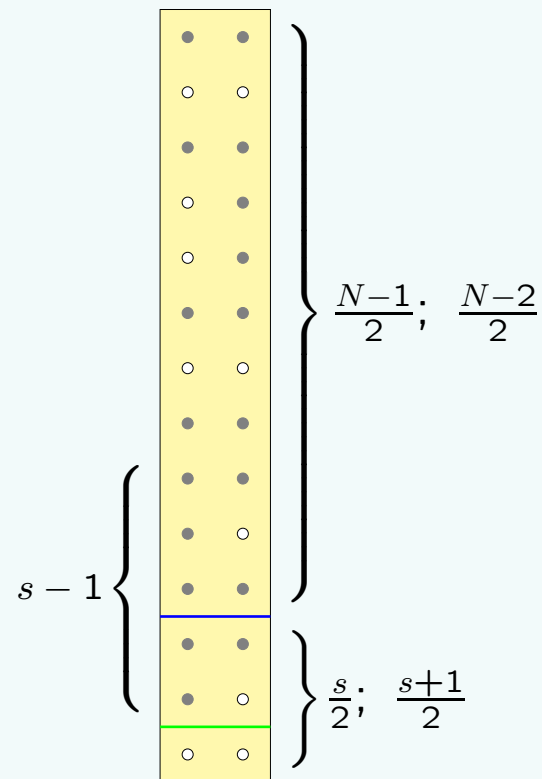
Example ($s = 6, N = 23$)



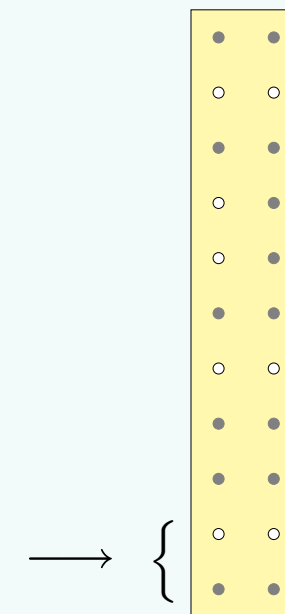
$$M = \frac{N+s-1}{2}, n = m$$

$$M = \frac{N-1}{2}; \frac{N-2}{2}, n = m + \frac{s-2}{2}; m + \frac{s-3}{2}, m + \frac{s-1}{2}$$

up	↔	left ●
valley	↔	left ○
down	↔	right ●
peak	↔	right ○



$$\begin{pmatrix} \bullet \\ \bullet \end{pmatrix} - \begin{pmatrix} \bullet \\ \circ \end{pmatrix} = \begin{pmatrix} \circ \\ \bullet \end{pmatrix}$$



← bijection →

← bijection →

-extends to (r, s)

Hamiltonian Limits

Expansion of $D(u)$

$$D(u) = I - 2uH + \mathcal{O}(u^2)$$

It follows that

$$-H = \left[\text{Diagram 1} \right] + \left[\text{Diagram 2} \right] + \dots + \left[\text{Diagram 3} \right]$$

The diagram shows the expansion of $-H$ as a sum of terms. Each term is a horizontal rectangle containing vertical lines and blue arcs. The first term has two blue arcs on the left side. The second term has two blue arcs on the right side. The third term has two blue arcs on the right side, with a different configuration. Ellipses indicate intermediate terms.

- In terms of the generators of the *linear* TL algebra, this corresponds to

$$H = - \sum_{j=1}^{N-1} e_j$$

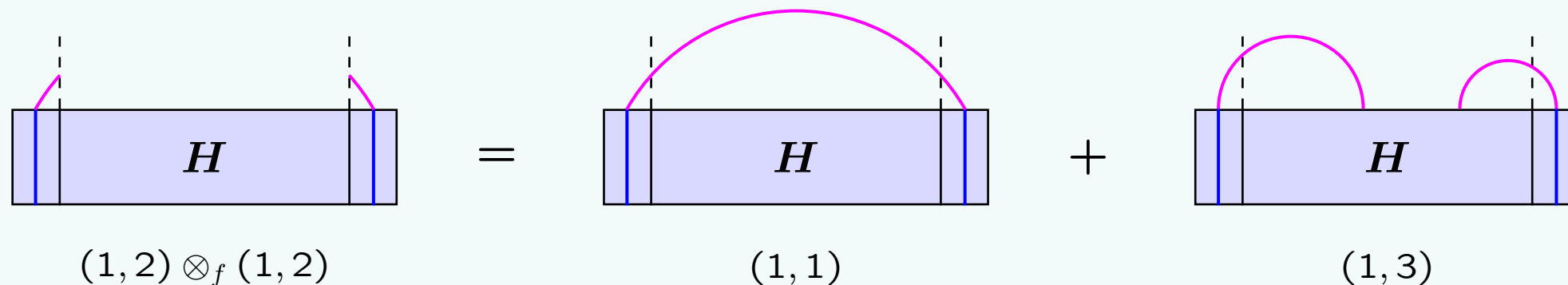
Defects Revisited



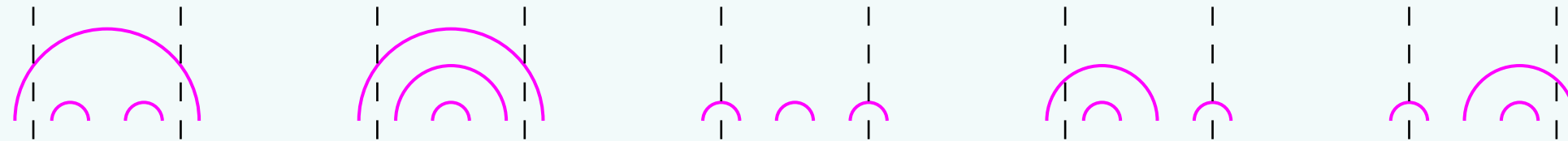
Fusion

-Diagrammatic Implementation-

Example: $(1, 2) \otimes_f (1, 2)$



● For $N = 4$, there are 5 link states:



Jordan Decomposition

$$\mathcal{H}_{(1,2)|(1,2)} = - \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \sqrt{2}I \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{8} & 1 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{pmatrix}$$

- Jordan cells are present for the **smallest** system sizes which can accommodate them.
- If a Jordan cell corresponding to a particular energy level is present for a particular system size, a Jordan cell corresponding to the **same** energy level is present for **all** larger system sizes.

Finitized Dilatation Generator

$$\mathcal{H}_{(1,2)|(1,2)} \longrightarrow L_0^{(1,2)|(1,2)} = \text{diag} \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right]$$

Finitized Partition Function

$$Z_{(1,2)|(1,2)}^{(4)}(q) = \chi_{(1,1)}^{(4)}(q) + \chi_{(1,3)}^{(4)}(q) = q^{1/12}[(1 + q^2) + (1 + q + q^2)] = q^{1/12}(2 + q + 2q^2)$$

Fusion Rules

$$(1, 2) \otimes_f (1, 2) = (1, 1) \oplus_i (1, 3) = \mathcal{R}_{1,1}$$

and likewise

$$(1, 2) \otimes_f (1, 3) = (1, 2) \oplus (1, 4)$$

$$(1, 2) \otimes_f \mathcal{R}_{1,1} = (1, 2) \oplus (1, 2) \oplus (1, 4)$$

$$\mathcal{R}_{1,1} \otimes_f \mathcal{R}_{1,1} = \mathcal{R}_{1,1} \oplus \mathcal{R}_{1,1} \oplus \mathcal{R}_{2,1}$$

⋮



Fusion Algebra

Concluding Remarks

Summary

- Exactly solvable lattice model for critical dense polymers
- Inversion identity
- Finite-size corrections
- CFT with spectrum $c = -2$, $\Delta = \Delta_{1,s}$, $s = 1, 2, 3, \dots$
- Physical combinatorics
- Link to Dyck paths
- Hamiltonian limits
- Diagrammatic implementation of fusion
- Jordan cells

Outlook

- Other geometries: cylinder and torus
- Other models: critical percolation, logarithmic Ising model, . . .