Sinh-Gordon Boundary TBA and Liouville Boundary Reflection Amplitude

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Outline of Talk

Boundary sinh-Gordon model is simple but has enough variety.

1. Introduction: bulk theory
   Sinh-Gordon as an integrable theory:
   Sinh-Gordon as a Liouville perturbation
2. Boundary Theory I
   Symmetric potential at one edge
3. Boundary Theory II
   Non-symmetric potential at both edges

Introduction

- What is a useful relation between the Sinh-Gordon Theory and Liouville Theory ?

(1) ShG model is integrable. (IR property)

Thermodynamic Bethe ansatz (TBA) allows to evaluate the ground energy at a given finite size.

(2) ShG model can be viewed as a perturbation

However, perturbing term is not compact: $\cosh(2b\phi)$

Liouville CFT (UV limit) + relevant perturbation.

Liouville theory provides other tool such as Liouville reflection amplitude (LRA) to find the same ground energy.

- The Two methods provide an independent check.

BASIC FACTS OF SINH-GORDON FEILD THEORY

ShG on an infinite plane (1)

- ShG action :

$$S_{\text{shG}} = \iint_{-\infty}^{\infty} dx dy \left[\frac{1}{4\pi} \left(\partial_a \phi \right)^2 + 2\mu \cosh \left(2b\phi \right) \right]$$

(Conveniently put on a Euclidean space-time)

- Non-trivial conservation laws ensure the integrability of the model. $\bar{\partial} T_{s+1} = \partial \Theta_{s-1}$
- There exists a single species of particle with a physical mass m

$$\pi \mu \gamma(b^2) = \left[\frac{m}{8\sqrt{\pi}} p^p (1-p)^{1-p} \Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{1-p}{2}\right) \right]^{2+2b^2}$$

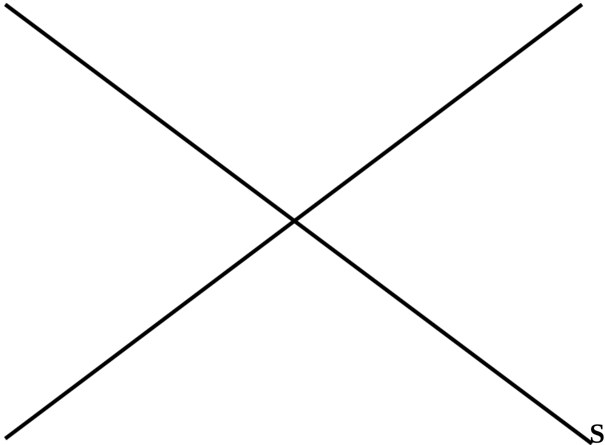
$$p = \frac{b^2}{1+b^2}$$

ShG on an infinite plane (2)

- Bulk vacuum energy: $\mathcal{E} = \frac{m^2}{8\sin\pi p}$
- scattering amplitude of the particle with mass m is given:

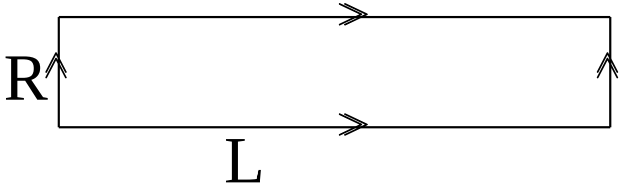
$$S(\theta) = \frac{\sinh\theta - i\sin\pi p}{\sinh\theta + i\sin\pi p}$$

No pole in the physical strip: $0 < \ \mathrm{Im}\,\theta < \pi$



Bulk TBA (1)

- Partition function: $Z = \mathrm{Tr}\, e^{-LH} = \mathrm{Tr}\, e^{-RH}$



- As $L \to \infty$, pick up the ground state energy E_0:

$$e^{-LE_0} = \mathrm{Tr}\, e^{-RH} = \sum_{\{\alpha\}} e^{-RE_\alpha}$$

(1) Introduce the spectral density (occupied and unoccupied) and write the free energy in terms of the spectral density

(2) Minimize the free energy to obtain the ground energy using the relation of spectral density through the Bethe ansatz equation.

Bulk TBA (2)

- Hamiltonian eigenstates are massive particle states, particle mass m and its energy $m\cosh\theta$. Treat the particles as obeying exclusion principle.

(1) Use spectral density $\rho(\theta)$ (particle density per length (L) per rapidity (θ)) to put free energy:

$$
\begin{aligned}
Z &= \int [d\rho] \exp(-RF) \\
F &= -\mathcal{S} + L \int_{-\infty}^{\infty} d\theta \rho(\theta) m \cosh\theta
\end{aligned}
$$

Bulk TBA (3)

(2) Entropy $\mathcal{S}$ counts the number of ways how to occupy the states (degeneracy of filling the states).

$$e^{\mathcal{S}} = \prod \frac{((\rho + \rho_h)L\Delta\theta)!}{(\rho L\Delta\theta)!(\rho_h L\Delta\theta)!}$$

$\rho_h(\theta)$ is the “hole" spectral function, representing unoccupied states.

(3) In the thermodynamic limit, $L \to \infty$, we may put the entropy as

$$\mathcal{S} = L \int_{-\infty}^{\infty} d\theta \, \{(\rho + \rho_h)\ln(\rho + \rho_h) - \rho \ln \rho - \rho_h \ln \rho_h\}$$

Bulk TBA (4)

(4) Minimize the free energy with respect to ρ and ρ_h.

$$\delta F = L \int_{-\infty}^{\infty} d\theta [\delta\rho \{ mR \cosh\theta - \ln(\frac{\rho+\rho_h}{\rho})\} - \delta\rho_h \{\ln(\frac{\rho+\rho_h}{\rho_h})\}] = 0$$

(5) Use the relation between spectral densities:

$$\exp(iLm \sinh\theta_i) \prod_{j=1,\neq i}^{N} S(\theta_i - \theta_j) = \exp(i2\pi n_i(\theta)) \,.$$

$$m \cosh\theta + \int_{-\infty}^{\infty} d\theta' \phi(\theta-\theta')\rho(\theta') = 2\pi(\rho(\theta)+\rho_h(\theta)) \,,$$

where $\phi(\theta) = -i\,\partial \ln S(\theta)/\partial\theta$.

$$n(\theta) = n_{\text{occupied}}(\theta) + n_{\text{unoccupied}}(\theta)$$

$$\rho(\theta) = \frac{1}{L}\frac{\partial n_{\text{occupied}}(\theta)}{\partial\theta} \,, \quad \rho_h(\theta) = \frac{1}{L}\frac{\partial n_{\text{unoccupied}}(\theta)}{\partial\theta} \,.$$

Bulk TBA (5)

- Ground energy is given as

$$E_0(R) = \mathcal{E}R - \frac{m}{2\pi}\int \cosh\theta \log\left(1 + e^{-\varepsilon(\theta)}\right) d\theta$$

$$\varphi(\theta) = -\frac{i}{2\pi}\frac{d}{d\theta}\log S(\theta) = \frac{1}{2\pi}\frac{4\sin\pi p\cosh\theta}{\cosh 2\theta - \cos 2\pi p}$$

- TBA for pseudo energy $\varepsilon(\theta) = \log(\rho_h/\rho)$:

$$\varepsilon(\theta) = mR\cosh\theta - \varphi * \log\left(1 + e^{-\varepsilon(\theta)}\right)$$

- bulk energy $\mathcal{E} = \frac{m^2}{8\sin\pi p}$ is added to normalize the energy

BASIC FACTS OF LIOUVILLE FIELD THEORY

Liouville on a cylinder (1)

- Liouville action on a cylinder with radius 2π:

$$S_{\mathrm{L}} = \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \left[\frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right]$$

(1) Add a background charge $Q = b + 1/b$ to make $\phi(z = \sigma + i\tau,\ \bar{z} = \sigma - i\tau)$ transform as:

$$\phi(w, \bar{w}) = \phi(z, \bar{z}) - \frac{Q}{2} \log \left| \frac{dw}{dz} \right|^2$$

(2) Energy-momentum tensor $T(z) = -(\partial\phi)^2 + Q\partial^2\phi$ ensures conformal invariance with central charge

$$c_{\mathrm{L}} = 1 + 6Q^2$$

Liouville on a cylinder (2)

(3) vertex operator $V_\alpha(x) = e^{2\alpha\phi}$ has a conformal dimension

$$\Delta(\alpha) = \alpha(Q-\alpha)\,,\,\bar{\Delta}(\alpha) = \alpha(Q-\alpha)\,.$$

2-point correlations are given as

$$\langle V_\alpha(x) V_{Q-\alpha}(y)\rangle = \frac{1}{|x-y|^{4\Delta(\alpha)}}$$

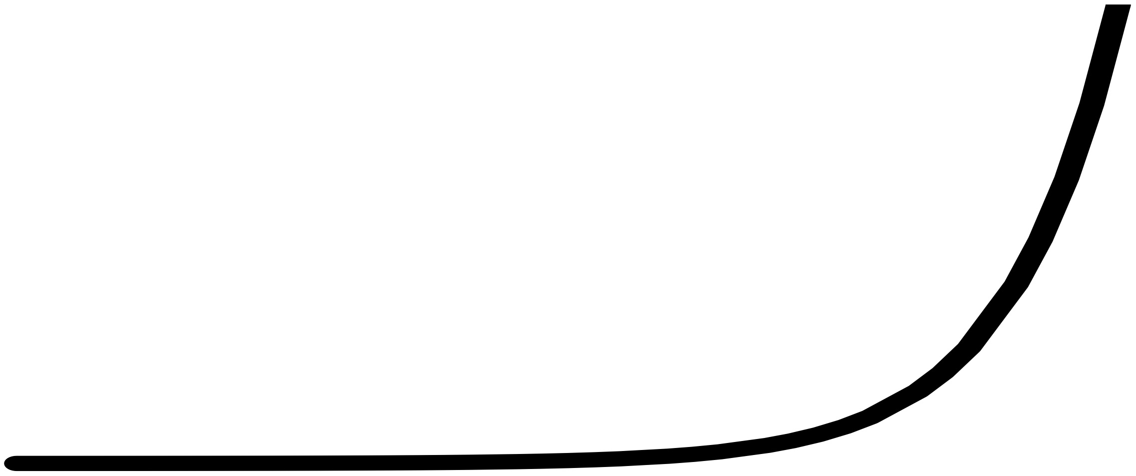
$$\langle V_\alpha(x) V_\alpha(y)\rangle = \frac{D(\alpha)}{|x-y|^{4\Delta(\alpha)}}$$

$D(\alpha)$ is called 2-point function.

Liouville on a cylinder (3)

- Louville potential and reflection amplitude

(1) Louville theory may be viewed as a free theory if the field far away from the wall $\exp(2b\phi)$



Liouville on a cylinder (4)

(2) In the flat region, ϕ is decomposed in oscillator modes :

$$\phi(z) = \phi_0 - \mathcal{P}(z - \bar{z}) + \sum_{n\neq 0} \left(\frac{ia_n}{n} e^{inz} + \frac{i\bar{a}_n}{n} e^{-inz} \right)$$

$$[a_m, a_n] = \frac{m\delta_{m+n}}{2}, \qquad [\bar{a}_m, \bar{a}_n] = \frac{m\delta_{m+n}}{2}$$

(4) ϕ_0 is the “zero mode” of the Liouville field

$$\phi_0 = \int_0^{2\pi} \phi(\sigma) \frac{d\sigma}{2\pi}, \qquad \mathcal{P} = -\frac{i}{2} \frac{\partial}{\partial \phi_0}$$

(5) Virasoro generators (holomorphic part):

$$L_0 = 2 \sum_{k>0} a_{-k} a_k + Q^2/4 + \mathcal{P}^2$$

$$L_n = \sum_{k\neq 0,n} a_k a_{n-k} + (2\mathcal{P} + inQ) a_n \qquad n \neq 0$$

Liouville on a cylinder (5)

(6) Primary state Ψ_P ($\alpha = Q/2 \pm iP\,;\ \ \Delta_P = Q^2/4 + P^2$) has the form at away from the wall:

$$\Psi_P \propto \left(e^{2iP\phi_0} + S_{\rm L}(P)e^{-2iP\phi_0}\right) |\text{Fock vacuum}\rangle$$

$S_{\rm L}(P)$ is called the Liouville Reflection Amplitude (LRA).

Liouville on a cylinder (6)

(7) Relation between $D(\alpha)$ and $S_{\mathrm{L}}(P)$:

$$D(\frac{Q}{2}+iP)=S_{\mathrm{L}}(P)$$

Explicit form of LRA

$$S_{\mathrm{L}}(P)=-\left(\pi\mu\gamma(b^2)\right)^{-2iP/b} \times \frac{\Gamma(1+2ibP)\Gamma(1+2iP/b)}{\Gamma(1-2ibP)\Gamma(1-2iP/b)}$$

duality $b\longrightarrow 1/b: \pi\mu_d\gamma(1/b^2)=\left(\pi\mu\gamma(b^2)\right)^{1/b^2}$

- One may check LRA using equation of motion at the classical limit $b\to 0$.

SHG AS A LIOUVILLE PERTURBATION THEORY

ShG as a Liouville perturbation (1)

- Put ShG field on a circle of length 2π (rescale $R \to R/(2\pi)$)

$$S_{\text{shG}} = \int dy \int_0^{2\pi} \Big[\frac{1}{4\pi} (\partial_a \phi)^2 + \mu \left(\frac{R}{2\pi} \right)^{2+2b^2} e^{2b\phi} \\ + \mu \left(\frac{R}{2\pi} \right)^{2+2b^2} e^{-2b\phi} \Big]$$

- Consider the zero mode ϕ_0 of Liouville field on a circle of length 2π. As $b\phi_0 \gg 1$, one may regard $V_{-b}(x) = e^{-2b\phi}$ as a perturbed potential:

ShG = Liouville + perturbation

ShG as a Liouville perturbation (2)

- As $\phi_0 \longrightarrow \infty$

(1) one has $S_{\text{shG}} = S_{\text{L}}\Big|_{\mu\to\tilde{\mu}} + (1,3)\ \text{perturbation}\,,$

$$\tilde{\mu} = \mu(R/2\pi)^{2+2b^2}$$

(2) Away from the Liouville potential wall $\exp(2b\phi)$, a primary state Ψ_P (dimension $\Delta_P = Q^2/4 + P^2$) has the asymptotic form

$$\Psi_P = \left(A_+ e^{2iP\phi_0} + A_- e^{-2iP\phi_0}\right) |\text{Fock vacuum}\rangle$$

$$\frac{A_-}{A_+} = \tilde{S}_{\text{L}}(P) = S_{\text{L}}(P)\Big|_{\mu\to\tilde{\mu}}$$

ShG as a Liouville perturbation (3)

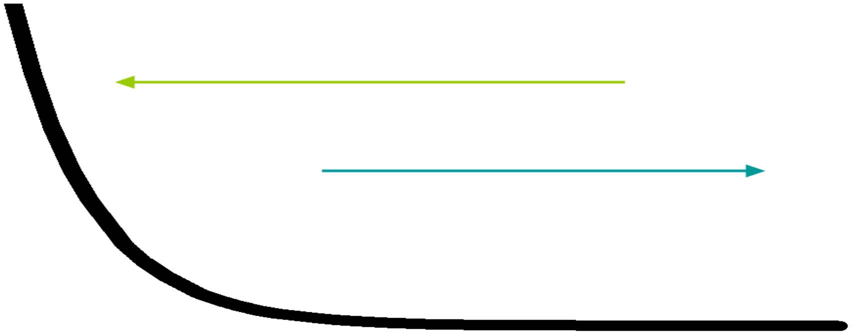
- Likewise, as $b\phi_0 \ll -1$

(1) Regard $V_b(x) = e^{2b\phi}$ as a perturbed term.

(2) The primary state away from the wall

$$\Psi_P = \left(A_+ e^{2iP\phi_0} + A_- e^{-2iP\phi_0}\right) |\text{Fock vacuum}\rangle$$

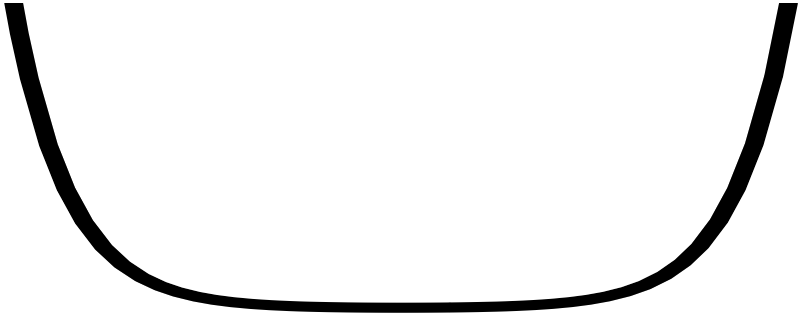
$$\frac{A_+}{A_-} = \tilde{S}_{\mathrm{L}}(P) = S_{\mathrm{L}}(P)\Big|_{\mu \to \tilde{\mu}}$$



ShG as a Liouville perturbation (4)

(3) Consistency requires P is quantized:

$$\left[\tilde{S}_{\mathrm{L}}(P)\right]^2 = 1\,; \quad \text{or} \quad \left[S_{\mathrm{L}}(P)\, e^{-4iPQ\,\log(R/2\pi)}\right]^2 = 1$$



In “mini-superspace" approach, the wave is moving inside the left and right potential wall.

ShG as a Liouville perturbation (5)

- Ground energy is given as

$$\begin{aligned} E_0 &= -\frac{\pi}{6R}\left(c_{\rm L} - 24\Delta_P\right) + \text{ power-like correction} \\ &= -\frac{\pi}{6R}\left(1 - 24P^2\right) + \text{ power-like correction} \end{aligned}$$

- Quantization condition determines P,

$$\left[S_{\rm L}(P)\, e^{-4iPQ\log(R/2\pi)}\right]^2 = 1; \qquad S_{\rm L}(P) \equiv -e^{i\Delta_L(P)}$$

$$\to \Delta_L(P) = \pi + 4PQ\log(R/2\pi) = \delta_1 P + \delta_3 P^3 + \delta_5 P^5 \cdots$$

- UV behavior is given as

$$E_0 = -\frac{\pi}{6R}\left(1 - \frac{24\pi^2}{\ell^2} + \frac{48\pi^4\delta_3(b)}{\ell^5} + \cdots\right) + O\left(R^{2+2b^2}\right)$$

$$\ell = \delta_1(b) - 4PQ\log(R/2\pi)$$

ShG as a Liouville perturbation (6)

Numerics for $\Delta_{\mathrm{L}}(P)$ and consistency check.

- LRA result: $S_{\mathrm{L}}(P) \equiv -\exp(i\Delta_{\mathrm{L}}(P))$

$$\Delta_{\mathrm{L}}(P) = -i\log\left((\pi\mu\gamma(b^2))^{-2iP/b}\frac{\Gamma(1+2ibP)\Gamma(1+2iP/b)}{\Gamma(1-2ibP)\Gamma(1-2iP/b)}\right)$$

- TBA result:

$$\Delta_{\mathrm{L}}^{\mathrm{TBA}}(P) = \pi + 4P_{\mathrm{TBA}}\,Q\log\left(\frac{R}{2\pi}\right)$$

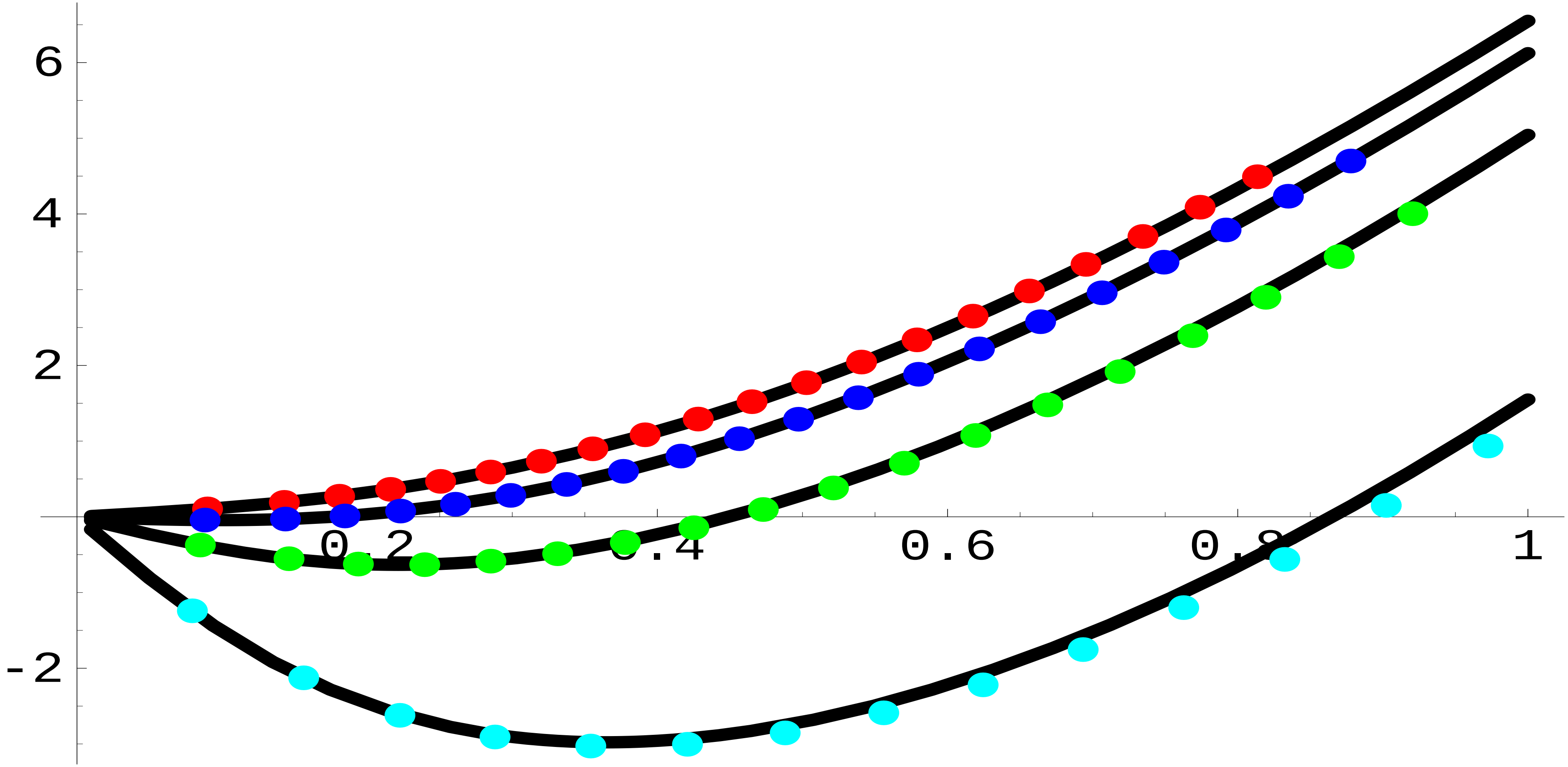
using $\quad P_{\mathrm{TBA}} = \sqrt{(1-c_{\mathrm{eff}}(R))/24} \quad$ and $\quad E_0^{\mathrm{TBA}}(R) = -\frac{\pi}{6R}c_{\mathrm{eff}(R)}$

$$E_0^{\mathrm{TBA}}(R) = -\frac{\pi}{6R}\left(1-24P_{\mathrm{TBA}}^2\right) = \mathcal{E}R - \frac{m}{2\pi}\int \cosh\theta\log\left(1+e^{-\varepsilon(\theta)}\right)d\theta$$

ShG as a Liouville perturbation (7)

Numerics of $\Delta_{\mathrm{L}}(P)$

$p = 0.4, p = 0.3, p = 0.2, p = 0.1$ from top to bottom



BOUNDARY THEORY I

SYMMETRIC POTENTIAL AT ONE EDGE

LIOUVILLE THEORY ON A STRIP

Liouville Theory on a strip (1)

- conformal boundary action (width = π)

$$S_{\text{strip}}(\tau) = \int_{-\infty}^{\infty} d\tau \left[\int_0^{\pi} \left(\frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right) d\sigma \right.$$
$$\left. + M_a e^{b\phi}(\tau, 0) + M_b e^{b\phi}(\tau, \pi) \right]$$

- boundary parameters are parametrized

$$M_{a,b} = \left(\frac{\mu}{\sin \pi b^2} \right)^{1/2} \cosh(\pi b s_{a,b})$$

$s_{a,b}$ is real or purely imaginary

(Use $s \to -s$ to restrict $0 \leq \text{Im}\, s \leq \frac{1}{b}$)

Liouville Theory on a strip (2)

- In the region $\phi_0 \to -\infty$,
  ϕ is decomposed in oscillator modes :

$$\phi(\sigma,\tau) = \phi_0 - 2i\mathcal{P}\tau + \sum_{n\neq 0} \frac{2ia_n}{n} e^{-n\tau} \cos(n\sigma) \ , \qquad [a_m, a_n] = \frac{m\delta_{m+n}}{2}$$

- ϕ_0 is the “zero mode” of the Liouville field

$$\phi_0 = \int_0^\pi \phi(\sigma) \frac{d\sigma}{\pi} \ , \qquad \mathcal{P} = -i\frac{\partial}{\partial \phi_0}$$

- Virasoro generator L_0

$$L_0 = 2\sum_{k>0} a_{-k}a_k + Q^2/4 + \mathcal{P}^2$$

Liouville Theory on a strip (3)

- Put background charge $Q = b + 1/b$ so that

(1) Bulk field $V_\alpha(x) = e^{2\alpha\phi}$ has a conformal dimension

$$\Delta(\alpha) = \alpha(Q-\alpha)\,, \qquad \bar{\Delta}(\alpha) = \alpha(Q-\alpha)\,.$$

(2) Boundary field $B^{\beta}_{s_a,s_b} = \left(e^{\beta\phi_B}\right)_{s_a,s_b}$ has a conformal dimension

$$\Delta_{\rm B}(\beta) = \beta(Q-\beta)\,.$$

- Boundary 2-point correlations are given as

$$\langle B^{\beta}_{s_a,s_b}(1)\ \ B^{Q-\alpha}_{s_b,s_a}(0)\rangle = 1 \quad \langle B^{\beta}_{s_a,s_b}(1)\, B^{\beta}_{s_b,s_a}(0)\rangle = D_{\rm B}(\beta)\,.$$

$D_{\rm B}(\beta)$ is called boundary 2-point function.

Liouville Theory on a strip (4)

- Primary state Ψ_P away from the wall with conformal dimension $\Delta = Q^2/4 + P^2$:

  $$\Psi_P \propto \left(\exp(iP\phi_0) + S_{\mathrm{B}}(P|s_a, s_b)\exp(-iP\phi_0)\right)|\text{Fock vac}$$

  $S_{\mathrm{B}}(P|s_a, s_b)$ is called the boundary LRA.

- Boundary LRA is given in terms of boundary 2-point function :

  $$D_{\mathrm{B}}(\beta) = S_{\mathrm{B}}(P|s_a, s_b)$$

  when $\beta = Q/2 \pm iP$.

Liouville Theory on a strip (5)

- Explicit form of boundary LRA:

$$S_{\mathrm{B}}(P|s_a,s_b) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{-iP/b} \frac{\Gamma_b(2iP)}{\Gamma_b(-2iP)} \times$$

$$\frac{S_b\left(\frac{Q}{2} - iP + i\frac{s_a+s_b}{2}\right) S_b\left(\frac{Q}{2} - iP - i\frac{s_a+s_b}{2}\right)}{S_b\left(\frac{Q}{2} + iP + i\frac{s_a-s_b}{2}\right) S_b\left(\frac{Q}{2} + iP - i\frac{s_a-s_b}{2}\right)}$$

- duality $b \to 1/b : s_i \to s_i \;\; (i=a,b)$

Liouville Theory on a strip (6)

Double-gamma function

- periodic property:

$$\Gamma_b(x+b) = \frac{(2\pi)^{1/2}}{b^{1/2-bx}\Gamma(bx)}\Gamma_b(x)\,, \quad \Gamma_b(x+1/b) = \frac{(2\pi)^{1/2}}{\Gamma(x/b)b^{x/b-1/2}}\Gamma_b(x)$$

- $\Gamma_b(x)$ has poles at
  $x = -mb - nb^{-1}$ (m and n non-negative integers)

- integral representation:

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t}\Big[\frac{e^{-xt}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{1}{t^2} - \frac{Q/2-x}{t} - \left(\frac{(x-Q/2)^2}{2} - \frac{b^2+b^{-2}}{24}\right)e^{-t}\Big]$$

Liouville Theory on a strip (7)

Double-sine function

- relation with double-gamma function

$$S_b(x) = \Gamma_b(x)/\Gamma_b(Q-x)\,, \qquad S_b(x)S_b(Q-x) = 1$$

- integral representation:

$$\log S_b(x) = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{dt}{t}\left[\frac{\sinh(Q-2x)t}{2\sinh(bt)\sinh(t/b)} - \frac{(Q/2-x)}{t}\right]$$

- meromorphic function of x : (m and n non-negative integers)

poles at $x = -mb - nb^{-1}$

zeros at $x = Q + mb + nb^{-1}$

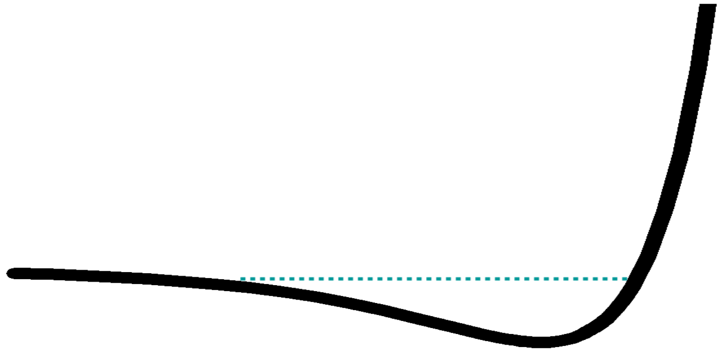
Liouville Theory on a strip (8)

- Thus BLRA can have pole(s) in an appropriate boundary parameter range.

$$S_{\mathrm{B}}(P|s_a, s_b) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{-iP/b} \frac{\Gamma_b(2iP)}{\Gamma_b(-2iP)} \times$$

$$\frac{S_b\left(\frac{Q}{2} - iP + i\frac{s_a+s_b}{2}\right) S_b\left(\frac{Q}{2} - iP - i\frac{s_a+s_b}{2}\right)}{S_b\left(\frac{Q}{2} + iP + i\frac{s_a-s_b}{2}\right) S_b\left(\frac{Q}{2} + iP - i\frac{s_a-s_b}{2}\right)}$$

- The pole refers to a bound state in the potential wall:



SINH-GORDON MODEL ON A HALF-LINE

ShG model on a half-line (1)

- Integrability ensures that the massive particle does not disappear on the edge but bounce back.
- Lagrangian

$$L_{\mathrm{shG}} = \int_{-\infty}^{0} \left(\frac{1}{4\pi} (\partial_a \phi)^2 + 2\mu \cosh(2b\phi) \right) dx$$
$$+ M^{+} e^{b\phi}(0) + M^{-} e^{-b\phi}(0)$$

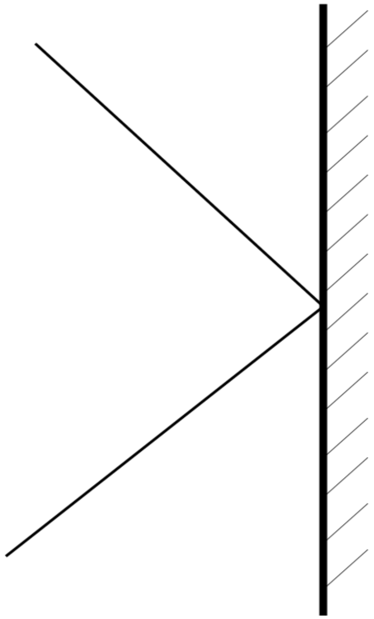
$$\text{(NOTE)} \quad M^{+} e^{b\phi} + M^{-} e^{-b\phi} = 2\mu_B \cosh(b\phi + b\phi_0)$$

$$M^{\pm} = \left(\frac{\mu}{\sin \pi b^2} \right)^{1/2} \cosh(\pi b s^{\pm})$$

ShG model on a half-line (2)

- boundary ShG model is integrable
- boundary scattering amplitude is defined as

$$A(\theta)B = R(\theta)A(-\theta)B$$



ShG model on a half-line (3)

- boundary scattering amplitude: $R(\theta) = R_0(\theta) R_1(\theta|\eta,\vartheta)$

$$R_0(\theta) = \frac{\sinh(\theta/2 + i\pi/4)}{\sinh(\theta/2 - i\pi/4)} \times \frac{\cosh(\theta/2 - i\pi p/4)}{\cosh(\theta/2 + i\pi p/4)} \frac{\cosh(\theta/2 - i\pi(1-p)/4)}{\cosh(\theta/2 + i\pi(1-p)/4)}$$

$$R_1(\theta|\eta,\vartheta) = \left(\frac{\sinh\theta - i\cosh(p\eta)}{\sinh\theta + i\cosh(p\eta)}\right)\left(\frac{\sinh\theta - i\cosh(p\vartheta)}{\sinh\theta + i\cosh(p\vartheta)}\right)$$

(η, ϑ will be restricted so that no pole in the physical strip: $0 <$ Im $\theta < \pi/2$)

- boundary parameters parametrization

$$\eta = \pi(s^+ + s^-)/(2b)\,; \qquad \vartheta = \pi(s^+ - s^-)/(2b)$$

$$M^\pm = \left(\mu/\sin(\pi b^2)\right)^{1/2} \cosh(\pi b s^\pm)$$

- R is invariant under the duality $b \longrightarrow 1/b$.

SINH-GORDON MODEL ON A STRIP

ShG on a strip (1)

- action (width R) : $A_{\text{strip}} = \int_{-\infty}^{\infty} L_{\text{strip}}(y)dy$

$$L_{\text{strip}}(y) = \int_0^R \left(\frac{1}{4\pi}(\partial_a\phi)^2 + 2\mu\cosh(2b\phi) \right) dx$$

$$+ M_1^+ e^{b\phi}(0) + M_1^- e^{-b\phi}(0) + M_2^+ e^{b\phi}(R) + M_2^- e^{-b\phi}(R)$$

- action (width π) (rescale $x = (R/\pi)\sigma$ and $y = (R/\pi)\tau$)

$$L_{\text{strip}}(\tau) = \int_0^\pi (\frac{1}{4\pi}(\partial_a\phi)^2 + 2\mu \left((R/\pi)^{2+2b^2} \cosh(2b\phi) \right) d\sigma$$

$$+ (R/\pi)^{1+b^2} \left(M_1^+ e^{b\phi}(0) + M_1^- e^{-b\phi}(0)\right.$$

$$\left. + M_2^+ e^{b\phi}(\pi) + M_2^- e^{-b\phi}(\pi)\right)$$

ShG on a strip (2)

- As $\phi_0 \to \infty$, one may neglect $e^{-2b\phi}$ in the bulk term and $e^{-b\phi}$ in the boundary term.
- Then the ShG reduces to the Liouville theory:

$$L_{\text{strip}}(\tau) = \int_0^\pi \left(\frac{1}{4\pi} (\partial_a \phi)^2 + \mu (R/\pi)^{2+2b^2} \exp(2b\phi) \right) d\sigma$$

$$+ (R/\pi)^{1+b^2} \left(M_1^+ e^{b\phi}(0) + M_2^+ e^{b\phi}(\pi) \right)$$

$\mu \to \mu (R/\pi)^{2+2b^2}$ but parameter relation is not affected.

- Primary field is given as

$$\Psi_0 \sim \left(A_+ \exp(iP\phi_0) + A_- \exp(-iP\phi_0) \right) |\text{Fock vacuum}\rangle$$

$$\frac{A_-}{A_+} = S_{\text{B}}(P|s_a^+, s_b^+)$$

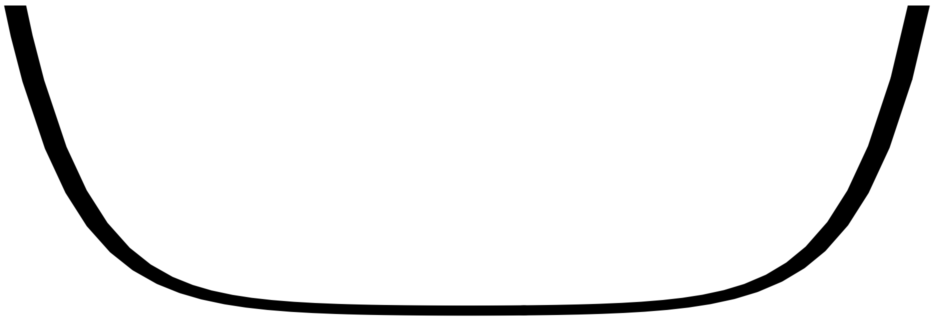
ShG on a strip (3)

- As $\phi_0 \to -\infty$, one has left wall.
- Thus ground energy is given as

$$E_0(R) = \frac{\pi}{R}\left(-\frac{1}{24} + P^2\right) + \text{power corrections of } R$$

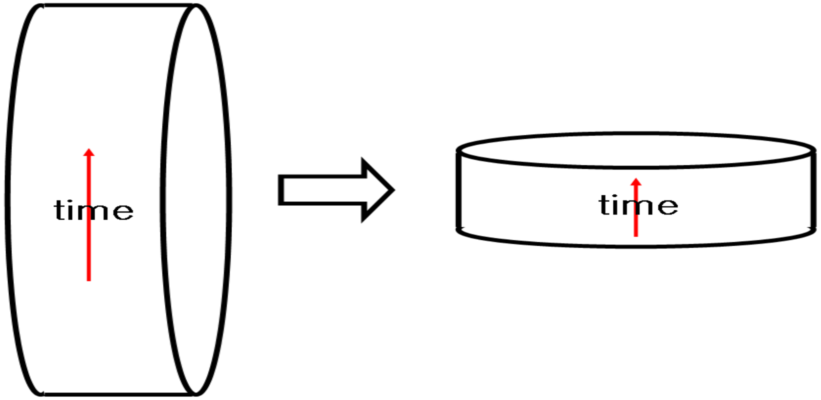
- And P is quantized:

$$(R/\pi)^{-4iPQ}\, S_{\mathrm{B}}(P|s_a^+, s_b^+)\, S_{\mathrm{B}}(P|s_a^-, s_b^-) = 1$$



ShG on a strip (4)

- The same ground energy can be obtained using the integrability of the theory: boundary TBA :
- Switch the role of time y and space x $\rightarrow$ “space" along y (infinite length L) and “time" along x (finite R)



- Initial state $|B1\rangle$ flows along R and arrives at final state $|B2\rangle$

ShG model on a strip (5)

- boundary TBA :

$$E_0(R) = \mathcal{E}R + f_a + f_b + E(R)$$

$$E(R) = -\frac{m}{4\pi}\int_{-\infty}^{\infty} \cosh\theta \log\left(1 + \lambda_{ab}(\theta)e^{-\varepsilon(\theta)}\right) d\theta$$

$$\varepsilon = 2mR\cosh\theta - \varphi * \log(1 + \lambda_{ab}e^{-\varepsilon})$$

- boundary energy

$$\begin{aligned} f(\eta,\vartheta) =& \frac{m}{4\sin(\pi p)}(2\cosh(p\eta) + 2\cosh(p\vartheta) \\ & - \sin(\pi p/2) - \cos(\pi p/2) - 1) \end{aligned}$$

ShG model on a strip (6)

- Boundary fugacity with $K_a(\theta) = R_a(i\pi/2 - \theta)$

$$\begin{aligned}
\lambda_{ab}(\theta) &= K_a(-\theta)K_b(\theta) \\
&= \coth^2\frac{\theta}{2}\cdot\frac{\cosh\theta+\cos\frac{\pi p}{2}}{\cosh\theta-\cos\frac{\pi p}{2}}\cdot\frac{\cosh\theta+\sin\frac{\pi p}{2}}{\cosh\theta-\sin\frac{\pi p}{2}} \\
&\times\frac{\cosh\theta-\cosh(\eta_a p)}{\cosh\theta+\cosh(\eta_a p)}\cdot\frac{\cosh\theta-\cosh(\eta_b p)}{\cosh\theta+\cosh(\eta_b p)} \\
&\times\frac{\cosh\theta-\cosh(\vartheta_a p)}{\cosh\theta+\cosh(\vartheta_a p)}\cdot\frac{\cosh\theta-\cosh(\vartheta_b p)}{\cosh\theta+\cosh(\vartheta_b p)}
\end{aligned}$$

- $\lambda_{ab}(\theta)$ is singular at $\theta = 0$ in general.
- boundary parameters: $M^{\pm} = (\mu/\sin(\pi b^2))^{1/2}\cosh(\pi b s^{\pm})$

$$\eta = \pi(s^+ + s^-)/(2b)\,; \qquad \vartheta = \pi(s^+ - s^-)/(2b)$$

SINH-GORDON MODEL ON A STRIP WITH SYMMETRIC POTENTIAL AT ONE EDGE

symmetric boundary at one edge (1)

- Suppose boundary potential is symmetric at a edge

$$M_a^+ = M_a^- \to \vartheta_a = 0\,, \qquad (\phi_0 = 0; \quad \mu_B \exp(b\phi + \phi_0))$$

- The fugacity becomes regular at $\theta = 0$:

$$\lambda_{ab}(\theta) = \frac{\cosh\theta + \cos(\pi p/2)}{\cosh\theta - \cos(\pi p/2)} \cdot \frac{\cosh\theta + \cos(\pi(1-p)/2)}{\cosh\theta - \cos(\pi(1-p)/2)} \times$$
$$\frac{\cosh\theta - \cosh(p\eta_a)}{\cosh\theta + \cosh(p\eta_a)} \cdot \frac{\cosh\theta - \cosh(p\eta_b)}{\cosh\theta + \cosh(p\eta_b)} \cdot \frac{\cosh\theta - \cosh\vartheta_b}{\cosh\theta + \cosh\vartheta_b}$$

(One may restrict $\vartheta_b \geq 0$
using the symmetry of action $\phi \to -\phi$ or $M^+ \leftrightarrow M^-$.)

- BTBA is not singular at real rapidity.

symmetric boundary at one edge (2)

Cross-check for the case with one symmetric edge

- One may check shG TBA result with the one from Liouville LRA within the proper boundary parameter range.
- M^+ and M^- have a lower bound for the theory to be well-defined:

$$M^\pm \geq -(\mu/\sin(\pi b^2))^{1/2}$$

symmetric boundary at one edge (3)

Summary for numerical check

- $\Delta(P|s_a, s_b)$ from Liouville Reflection Amplitude

(1) $S_{\mathrm{L}}(P) \equiv \exp(i\Delta_{\mathrm{L}}(P))\,, \quad S_{\mathrm{B}}(P|s_a^+, s_b^+) \equiv \exp(i\Delta_{\mathrm{B}}(P|s_a^-, s_b^-))$

(2) $\Delta(P|s_a, s_b) = \Delta_{\mathrm{B}}(P) - \frac{1}{2}\Delta_{\mathrm{L}}(P)$

$$\Delta(P|s_a, s_b) = \int\limits_{-\infty}^{\infty} \frac{\sin(2Pt)dt}{t} \times$$

$$\left(\frac{\cos(s_a t)\cos(s_b t) - \cosh(bt/2)\cosh(b^{-1}t/2)\cosh((b-b^{-1})t/2)}{\sinh(bt)\sinh(t/b)}\right)$$

symmetric boundary at one edge (4)

- $\Delta(P_{\rm TBA}|s_a, s_b)$ from TBA result

(1) Using the quantization relation:

$$\Delta_{\rm B}(P_{\rm TBA}|s_a^+, s_b^+) + \Delta_{\rm B}(P_{\rm TBA}|s_a^-, s_b^-) = 2\pi + 4PQ\log(R/\pi)$$

$$\frac{1}{2}(\Delta(P_{\rm TBA}|s_a^+, s_b^+)+\Delta(P_{\rm TBA}|s_a^-, s_b^-)) = \pi+2P_{\rm TBA}Q\log(R/\pi)+\Delta_{\rm L}(P_{\rm TBA})$$

(2) Find $P_{\rm TBA}$ using TBA $P_{\rm TBA} = \sqrt{(1-c_{\rm eff}(R))/24}$:

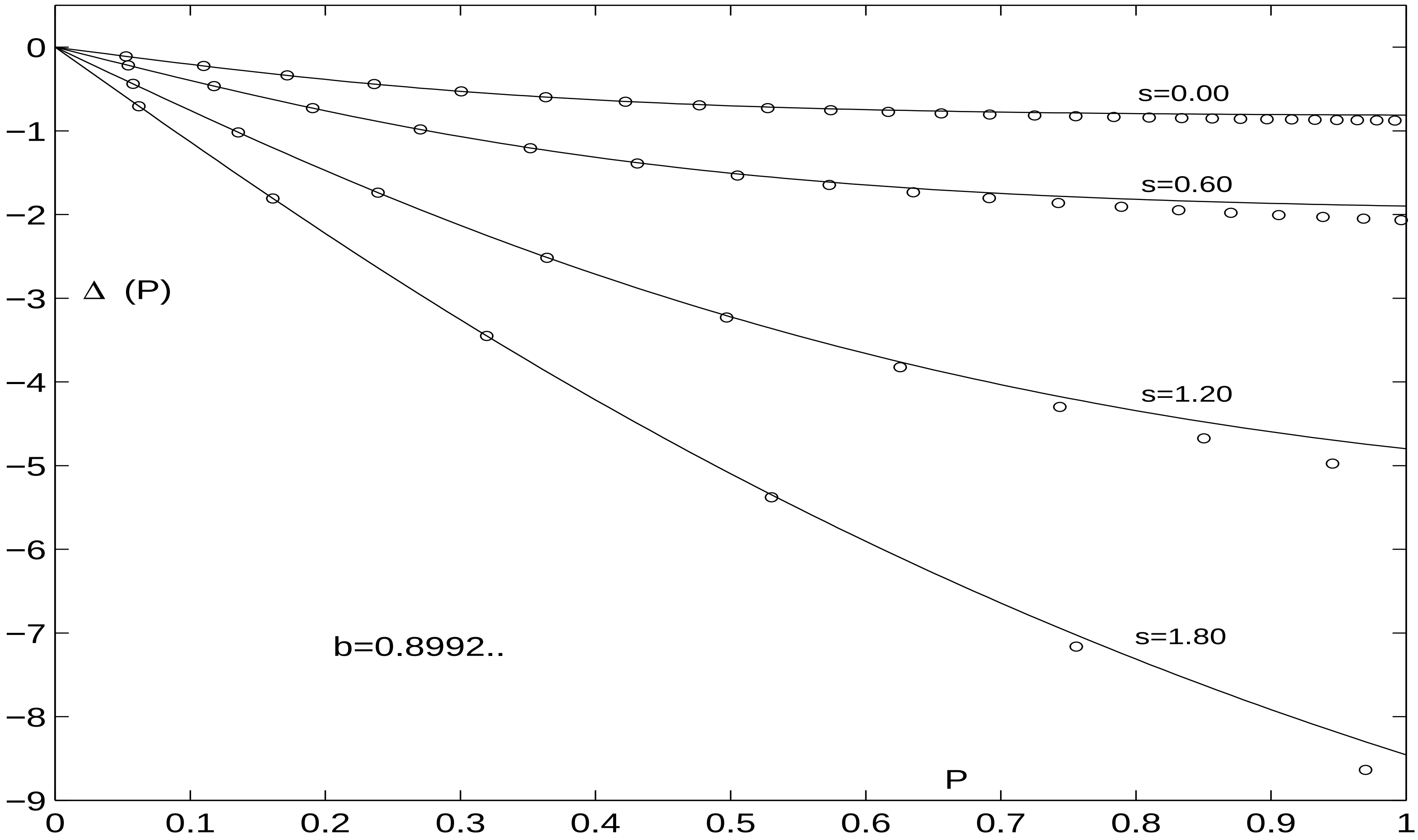
$$E_0(R) = -\frac{\pi}{24R}c_{{\rm eff}(R)} = \frac{\pi}{R}\left(-\frac{1}{24} + P_{\rm TBA}^2\right) = \mathcal{E}R + f_a + f_b + E(R)$$

$$E(R) = -\frac{m}{4\pi}\int_{-\infty}^{\infty}\cosh\theta\log\left(1+\lambda_{ab}(\theta)e^{-\varepsilon(\theta)}\right)d\theta$$

$$\varepsilon = 2mR\cosh\theta - \varphi * \log(1+\lambda_{ab}e^{-\varepsilon})$$

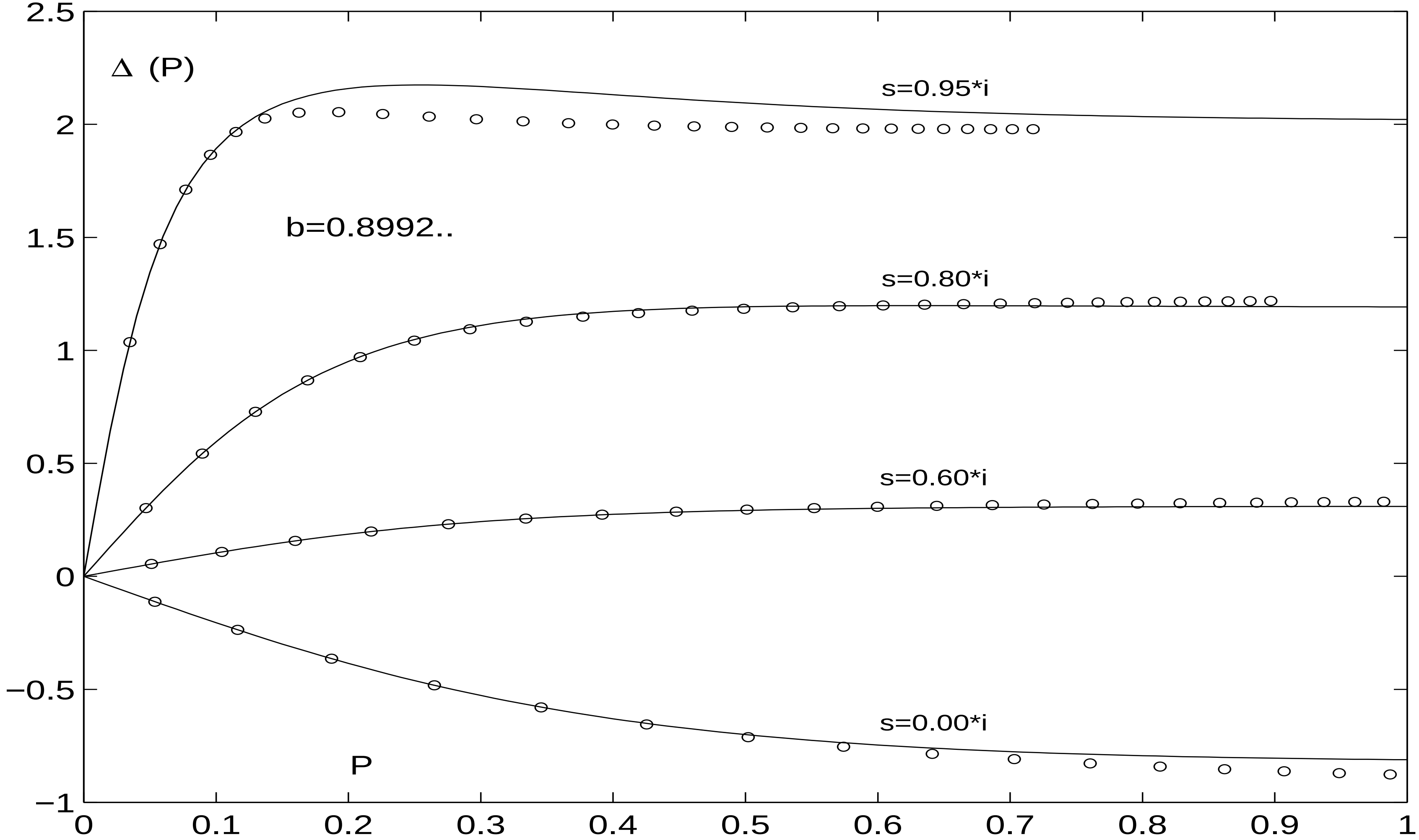
symmetric boundary at one edge (5)

$\Delta(P|s_a, s_b)$ for $b^2 = 0.8086$ and $s_a = s_b$ real



symmetric boundary at one edge (6)

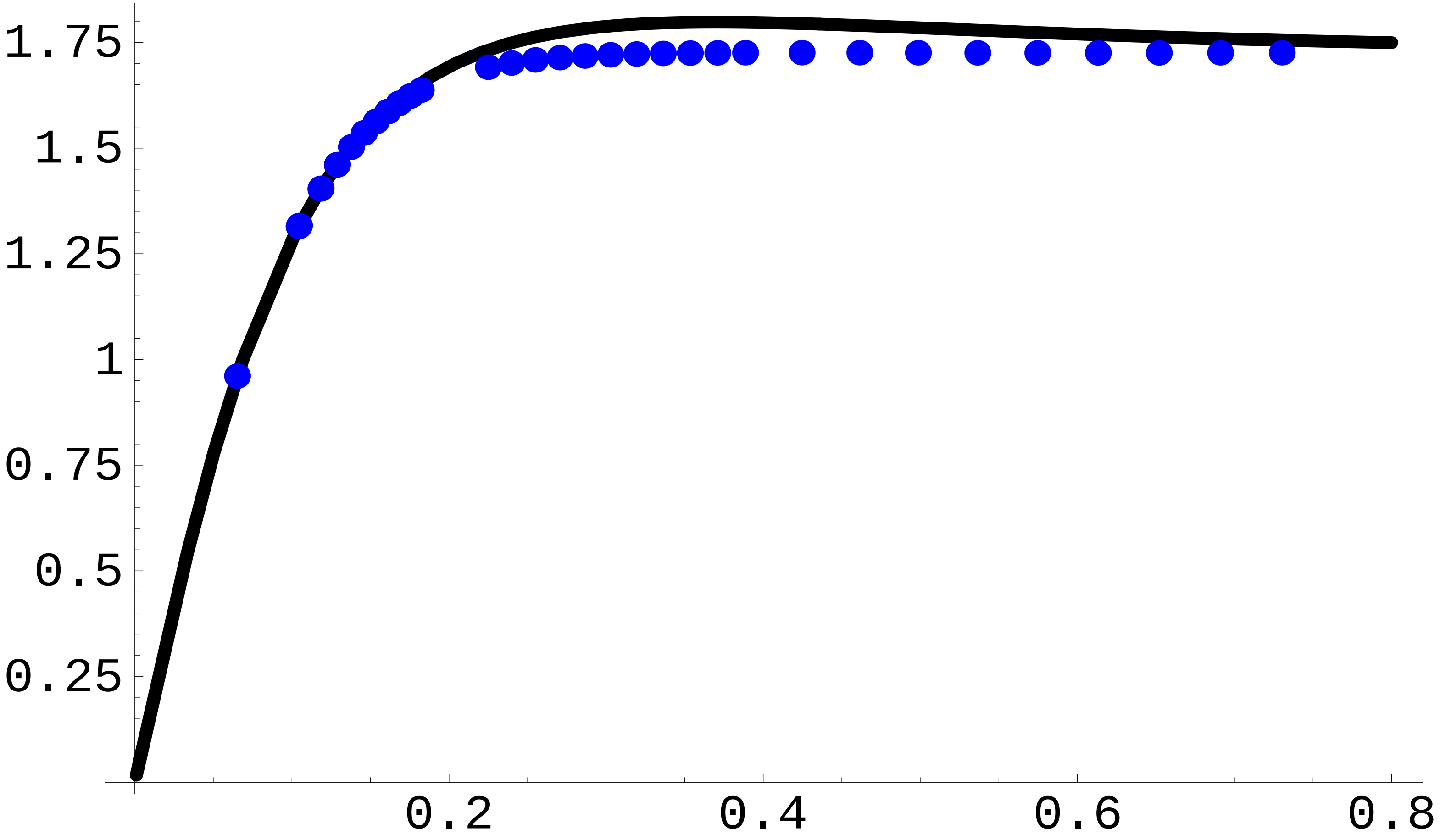
$\Delta(P|s_a, s_b)$ for $b^2 = 0.8086$ and $s_a = s_b$ imaginary



symmetric boundary at one edge (7)

$\Delta(P|s_a, s_b)$ for $b^2 = 2/3$ ($p = 0.4$) and

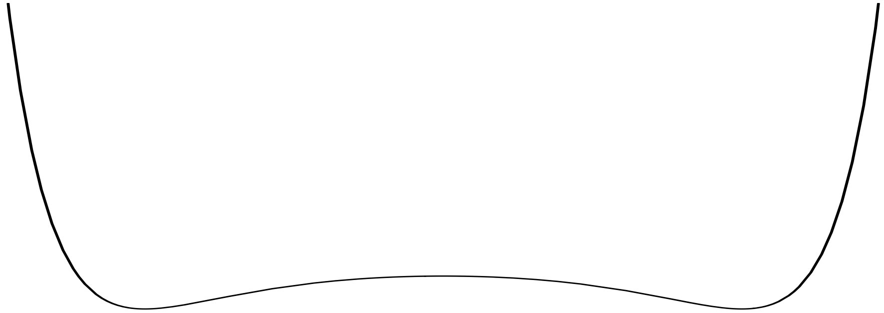
$\eta_a = \eta_b = 4.5i\pi/4 < 5i\pi/4$



symmetric boundary at one edge (8)

Unexpected phenomenon from Liouville bound state

- If $M_\pm$ is in a certain negative range, one (or more) bound state(s) may appear in the Liouville potential:

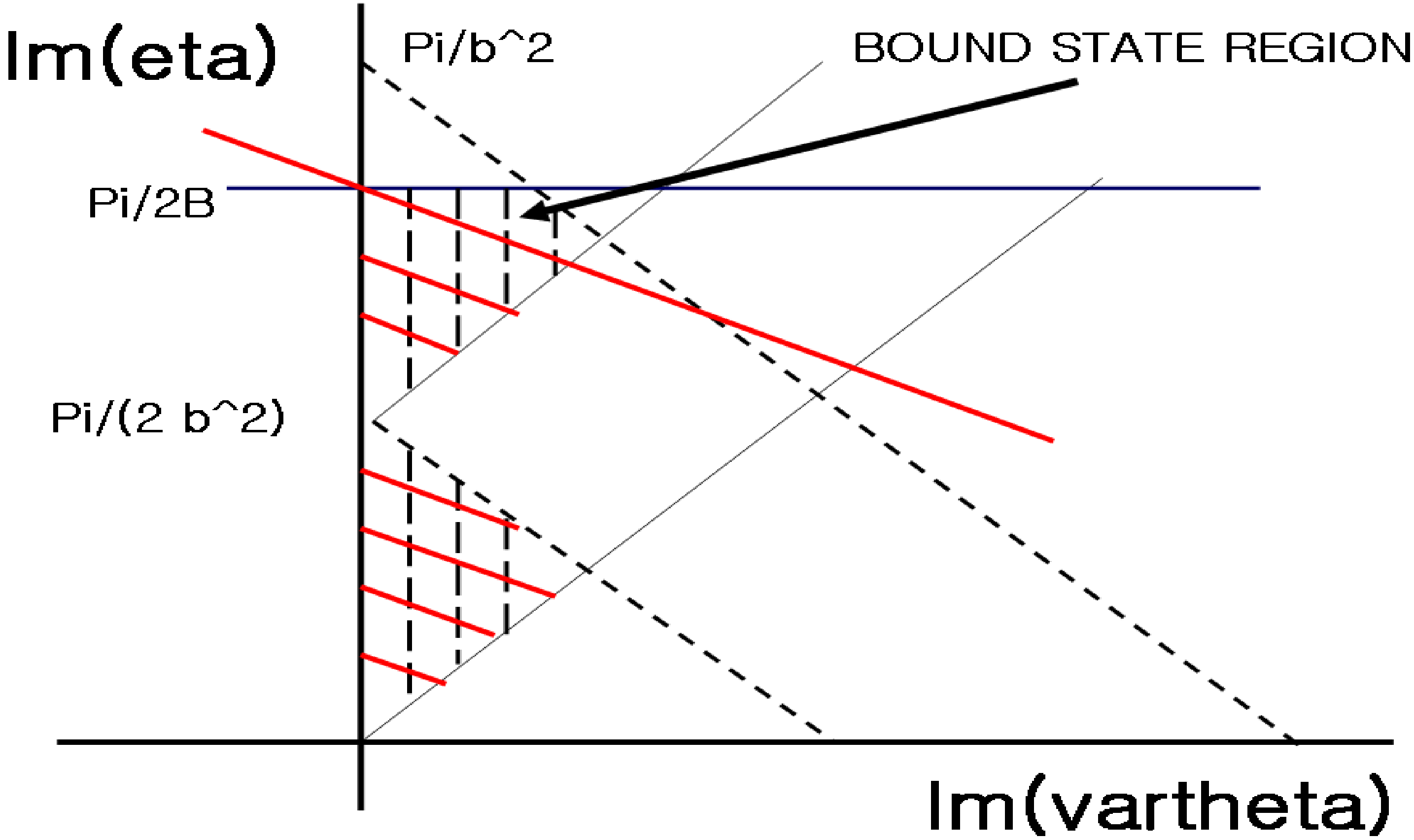


- The boundary state is given in terms of the pole position of S_B and P is imaginary :

$$E_0(R) = -\frac{\pi}{24R}\left(1 - 24P^2\right) < -\frac{\pi}{24R}$$

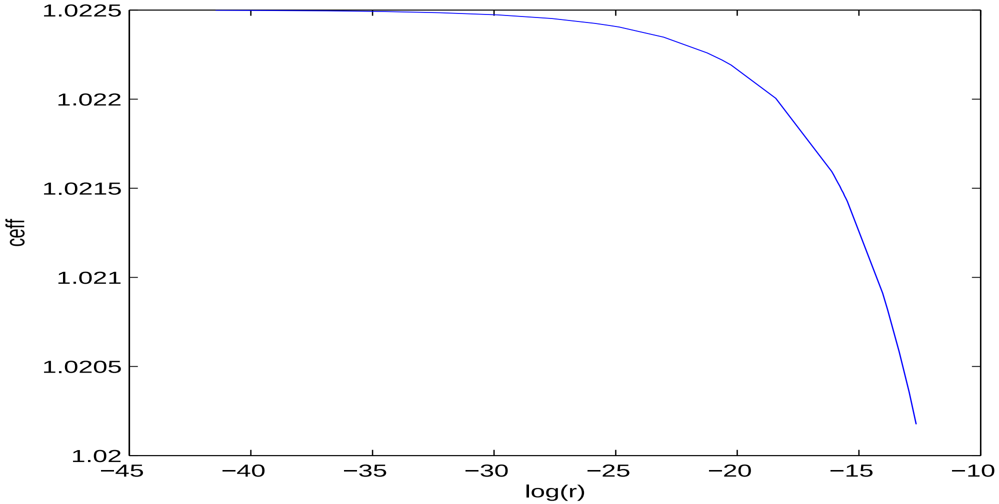
symmetric boundary at one edge (10)

Parameter Space Diagram
Case for $\vartheta_a = 0$ (one symmetric edge)
and $\eta_a = \eta_b = \eta$ and $\vartheta_b = \vartheta$



symmetric boundary at one edge (11)

- For parameter $B = 0.4$, $\eta_a = \eta_b = 9.5i\pi/8$, $\vartheta_a = 0$, $\vartheta_b = i\pi/5$, $bs_a^+ \cong 0.925i,\ bs_a^- \cong 0.658i,\ bs_b^+ = bs_b^- \cong 0.792i;\quad M_{a,b}^{\pm} < 0$
  BLRA predicts $c_{\text{eff}}(0) = 1 + 24\left(\frac{|s_a^+ + s_b^+| - Q}{2}\right)^2 = 1.0225 > 1$
  Here is the BTBA result:



- The UV ground state is not trivial, which is not related with boundary bound state(s).

BOUNDARY THEORY II

NON-SYMMETRIC POTENTIAL AT BOTH EDGES

Non-symmetric boundary (1)

- If both of boundaries do not satisfy $M^{+}_{(a,b)} \neq M^{-}_{(a,b)}$, $(\phi_0^{(a,b)} \neq 0)$ $\lambda_{ab}(\theta)$ is singular at $\theta = 0$.
- In this case, one may modify BTBA to have the correct numerics:

$$\begin{aligned}
\varepsilon &= 2mR\cosh\theta - \zeta(\theta) - \varphi * L_{\mathrm{s}}(\theta) \\
L_{\mathrm{s}}(\theta) &= \log\left(\tanh^2\theta + \tanh^2\theta\lambda_{\alpha\beta}(\theta)\, e^{-\epsilon(\theta)}\right) \\
\zeta(\theta) &= \varphi * \log(\coth^2\theta) \\
&= \log\left(\frac{\cosh\theta + \sin\pi p}{\cosh\theta - \sin\pi p}\right)\left(\frac{\cosh 2\theta + \cos 2\pi p}{\cosh 2\theta - \cos 2\pi p}\right)
\end{aligned}$$

Non-symmetric case (2)

- $\tanh^2\theta\ \lambda_{\alpha\beta}(\theta)$ is not singular at $\theta = 0$.

$$L_{\mathrm{s}}(\theta) = \log\left(\tanh^2\theta + \tanh^2\theta\lambda_{\alpha\beta}(\theta)\, e^{-\epsilon(\theta)}\right)$$

However, one needs be careful about numerics at a large distance.

Non-symmetric case (3)

- Another form : $\varepsilon = 2mR\cosh\theta - \tau(\theta) - \varphi * L_{\text{reg}}(\theta)$

$$L_{\text{reg}} = \log\left(\{1+\lambda_{ab}(\theta)e^{-\varepsilon}\}/\{1+g_\alpha^2 g_\beta^2 e^{-2mR}/(4\sinh^2\theta)\}\right)$$

$$4\theta^2\lambda_{\alpha\beta}(\theta)|_{\theta\to 0} = g_\alpha^2 g_\beta^2\,, \qquad 0 < \sin\gamma\pi \equiv |g_1 g_2|e^{-mR}/2 < 1$$

$$\begin{aligned}\tau(\theta) &= \varphi * \log\left(1+g_\alpha^2 g_\beta^2 e^{-2\ell}/(4\sinh^2\theta)\right)\\ &= \frac{1}{2}\ln\left\{\frac{\cosh\theta-\cos\pi(p+\gamma)}{\cosh\theta+\cos\pi(p+\gamma)}\;\frac{\cosh\theta+\cos\pi(p-\gamma)}{\cosh\theta-\cos\pi(p-\gamma)}\right.\\ &\qquad\left.\frac{(\cosh 2\theta-\cos 2\pi(p+\gamma))(\cosh 2\theta-\cos 2\pi(p-\gamma))}{(\cosh 2\theta-\cos 2\pi p)^2}\right\}\end{aligned}$$

- This form gives the long distance behavior explicitly:

$$E^0_{\alpha\beta}(R) = -m\frac{|g_\alpha g_\beta|}{4}e^{-mR} - \frac{m}{4\pi}\int_{-\infty}^{\infty} d\theta\;\cosh\theta\, L_{\text{reg}}(\theta)$$

Non-symmetric case (4)

proper boundary parameter range

- M^+ and M^- have a lower bound for the theory to be well-defined: $M^\pm \geq -(\mu/\sin(\pi b^2))^{1/2}$
- A certain negative $M^+_{(a,b)}$ and $M^-_{(a,b)}$ can result in a bound state: The pole position of $S_{\rm B}$ is given in terms of pure imaginary value P: Non-trivial UV vaccumm.
- In addition, there are two different cases to be distinguished: (1) $\vartheta_a \cdot \vartheta_b \geq 0$, (2) $\vartheta_a \cdot \vartheta_b < 0$
  BTBA is not sensitive to the signature of $\vartheta_a \cdot \vartheta_b$
  BLRA is sensitive to the signature of $\vartheta_a \cdot \vartheta_b$.

Non-symmetric case (5)

TBA is not sensitive to the sign of ϑ. fugacity is given as $\cosh(\vartheta p)$:

$$
\begin{aligned}
\lambda_{ab}(\theta) &= K_a(-\theta)K_b(\theta) \\
&= \coth^2\frac{\theta}{2}\cdot\frac{\cosh\theta+\cos\frac{\pi p}{2}}{\cosh\theta-\cos\frac{\pi p}{2}}\cdot\frac{\cosh\theta+\sin\frac{\pi p}{2}}{\cosh\theta-\sin\frac{\pi p}{2}} \\
&\times\frac{\cosh\theta-\cosh(\eta_a p)}{\cosh\theta+\cosh(\eta_a p)}\cdot\frac{\cosh\theta-\cosh(\eta_b p)}{\cosh\theta+\cosh(\eta_b p)} \\
&\times\frac{\cosh\theta-\cosh(\vartheta_a p)}{\cosh\theta+\cosh(\vartheta_a p)}\cdot\frac{\cosh\theta-\cosh(\vartheta_b p)}{\cosh\theta+\cosh(\vartheta_b p)}
\end{aligned}
$$

Non-symmetric case (6)

BLRA is sensitive to the signature of $\vartheta_a \cdot \vartheta_b$ or $\phi_0^{(a)} \cdot \phi_0^{(b)}$

- boundary parameters: $M^\pm = (\mu/\sin(\pi b^2))^{1/2} \cosh(\pi b s^\pm)$
  $\eta = \pi(s^+ + s^-)/(2b)\,; \qquad \vartheta = \pi(s^+ - s^-)/(2b)$
  $s^\pm >$ is chosen: $\vartheta \to -\vartheta$ results in $s^+ \leftrightarrow s^-$.
- case with $\vartheta_a \cdot \vartheta_b > 0$

$$e^{-4iPQ\log(L/\pi)}\, S_B(P|s_a^+, s_b^+)\, S_B(P|s_a^-, s_b^-) = 1$$

- case with $\vartheta_a \cdot \vartheta_b < 0$.

$$e^{-4iPQ\log(L/\pi)}\, S_B(P|s_a^+, s_b^-)\, S_B(P|s_a^-, s_b^+) = 1$$

Non-symmetric case (7)

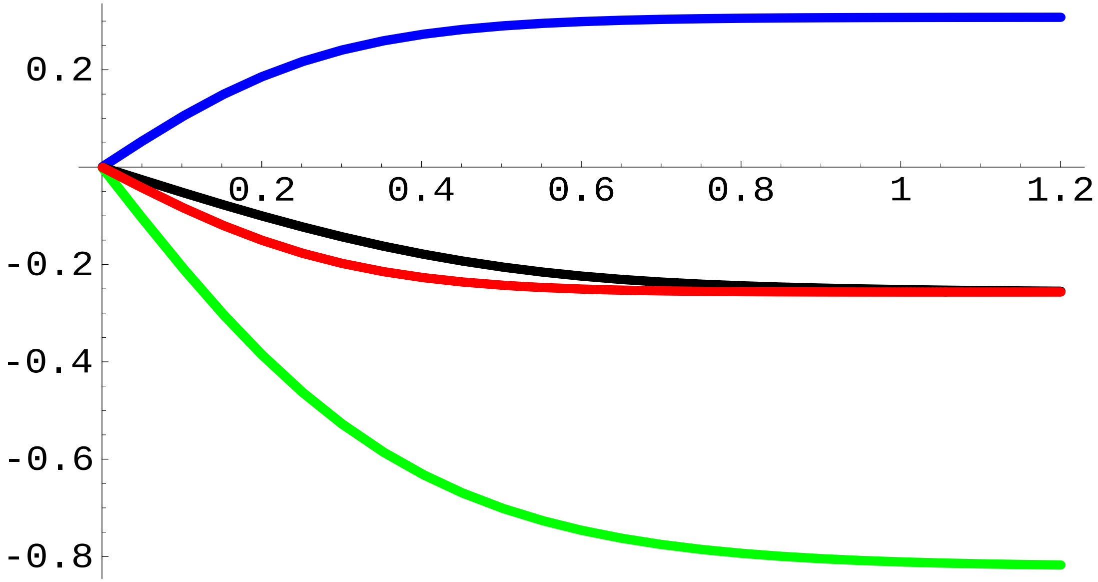
choose parameters

$$b^2 = 0.8086\,, \quad \eta_a = \eta_b = \vartheta_a = \vartheta_b = i\pi/3$$

$$s_a^+ = s_b^+ = \tfrac{b}{\pi}(\eta_a + \vartheta_a) = i\tfrac{2b}{3}\,, \; s_a^- = s_b^- = \tfrac{b}{\pi}(\eta_a - \vartheta_a) = 0$$

$\Delta_{\mathrm{B}}(P|s_a^+, s_b^+) = \Delta_{\mathrm{B}}(P|i\tfrac{2b}{3}, i\tfrac{2b}{3})$, $\Delta_{\mathrm{B}}(P|s_a^-, s_b^-) = \Delta_{\mathrm{B}}(0,0)$

$\left(\frac{\Delta_{\mathrm{B}}(P|s_a^+,s_b^+)+\Delta_{\mathrm{B}}(P|s_a^-,s_b^-)}{2}\right)$, $\Delta_{\mathrm{B}}(P|s_a^+, s_b^-) = \Delta_{\mathrm{B}}(P|i\tfrac{2b}{3}, 0)$



Non-symmetric case (8)

Numerics is done for $\Delta(P|s_a, s_b)$

- $S_{\mathrm{L}}(P) \equiv \exp(i\Delta_{\mathrm{L}}(P)) \quad S_{\mathrm{B}}(P|s_a^+, s_b^+) \equiv \exp(i\Delta_{\mathrm{B}}(P|s_a^-, s_b^-))$
- $\Delta(P|s_a, s_b) = \Delta_{\mathrm{B}}(P) - \frac{1}{2}\Delta_{\mathrm{L}}(P)$

$$\Delta(P|s_a, s_b) = \int_{-\infty}^{\infty} \frac{\sin(2Pt)dt}{t} \times$$

$$\left(\frac{\cos(s_a t)\cos(s_b t) - \cosh(bt/2)\cosh(b^{-1}t/2)\cosh((b-b^{-1})t/2)}{\sinh(bt)\sinh(t/b)}\right)$$

- TBA result using $P_{\mathrm{TBA}} = \sqrt{(1-c_{\mathrm{eff}})/24}$

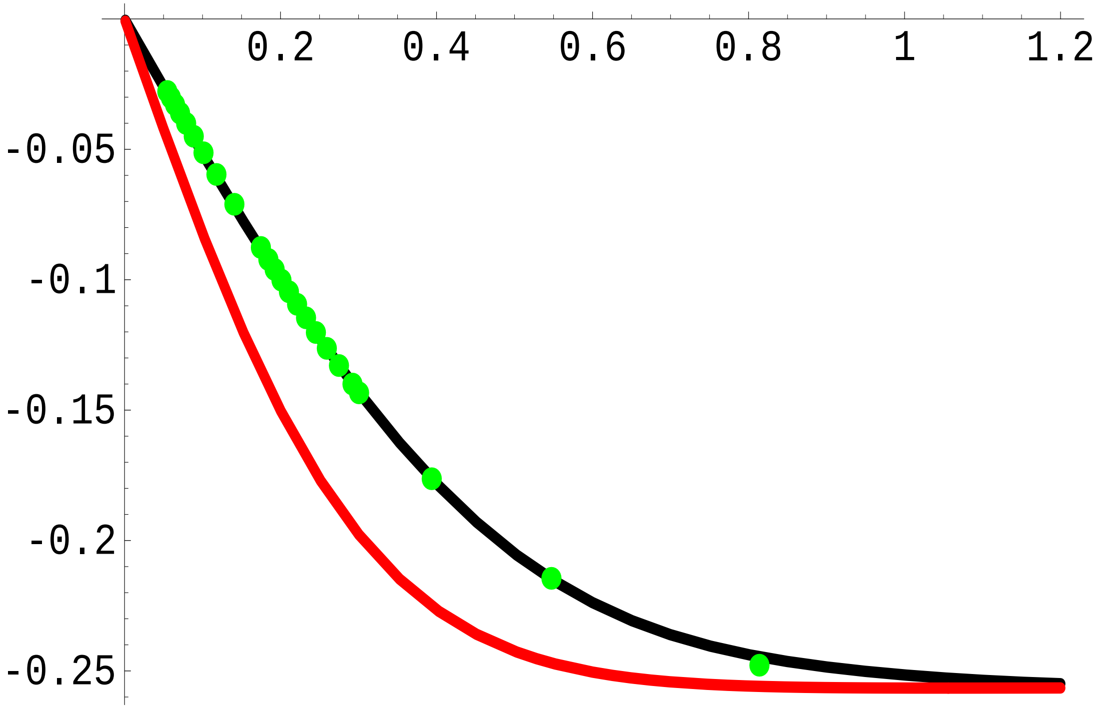
or $\quad E_0^{\mathrm{TBA}}(R) = -\frac{\pi}{24R}c_{\mathrm{eff}} = \frac{\pi}{R}\left(-\frac{1}{24} + P_{\mathrm{TBA}}^2\right)$

Non-symmetric case (9)

$$\text{TBA with } b^2 = 0.8086\,,\quad \eta_a = \eta_b = \vartheta_a = \vartheta_b = i\pi/3$$

$$s_a^+ = s_b^+ = \frac{b}{\pi}(\eta_a + \vartheta_a) = i\frac{2b}{3}\,,\ s_a^- = s_b^- = \frac{b}{\pi}(\eta_a - \vartheta_a) = 0$$

$$\left(\frac{\Delta(P|s_a^+,s_b^+)+\Delta(P|s_a^-,s_b^-)}{2}\right), \qquad \left(\frac{\Delta(P|s_a^+,s_b^-)+\Delta(P|s_a^-,s_b^+)}{2}\right)$$



Non-symmetric case (10)

Case with $\vartheta_a \cdot \vartheta_b < 0$

TBA is not sensitive to the sign of ϑ. fugacity is given as $\cosh(\vartheta p)$:

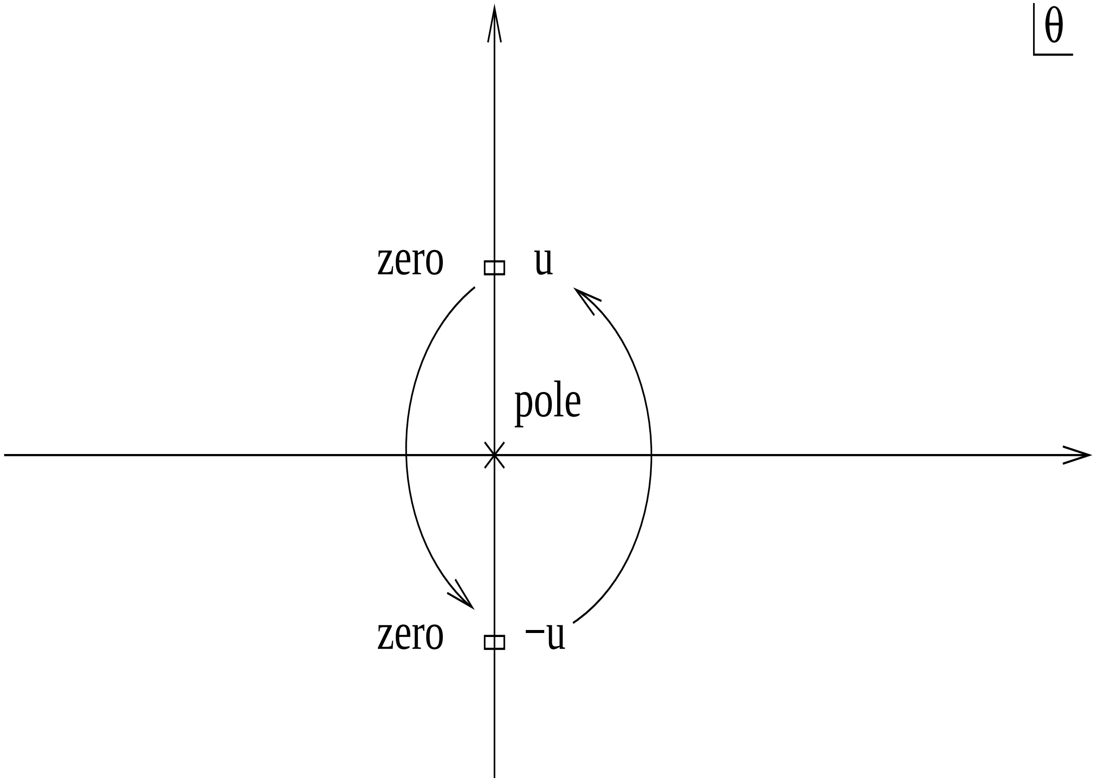
$$\begin{aligned}
\lambda_{ab}(\theta) &= K_a(-\theta)K_b(\theta) \\
&= \coth^2\frac{\theta}{2} \cdot \frac{\cosh\theta + \cos\frac{\pi p}{2}}{\cosh\theta - \cos\frac{\pi p}{2}} \cdot \frac{\cosh\theta + \sin\frac{\pi p}{2}}{\cosh\theta - \sin\frac{\pi p}{2}} \\
&\times \frac{\cosh\theta - \cosh(\eta_a p)}{\cosh\theta + \cosh(\eta_a p)} \cdot \frac{\cosh\theta - \cosh(\eta_b p)}{\cosh\theta + \cosh(\eta_b p)} \\
&\times \frac{\cosh\theta - \cosh(\vartheta_a p)}{\cosh\theta + \cosh(\vartheta_a p)} \cdot \frac{\cosh\theta - \cosh(\vartheta_b p)}{\cosh\theta + \cosh(\vartheta_b p)}
\end{aligned}$$

Non-symmetric case (11)

Case with $\vartheta_a \cdot \vartheta_b < 0$

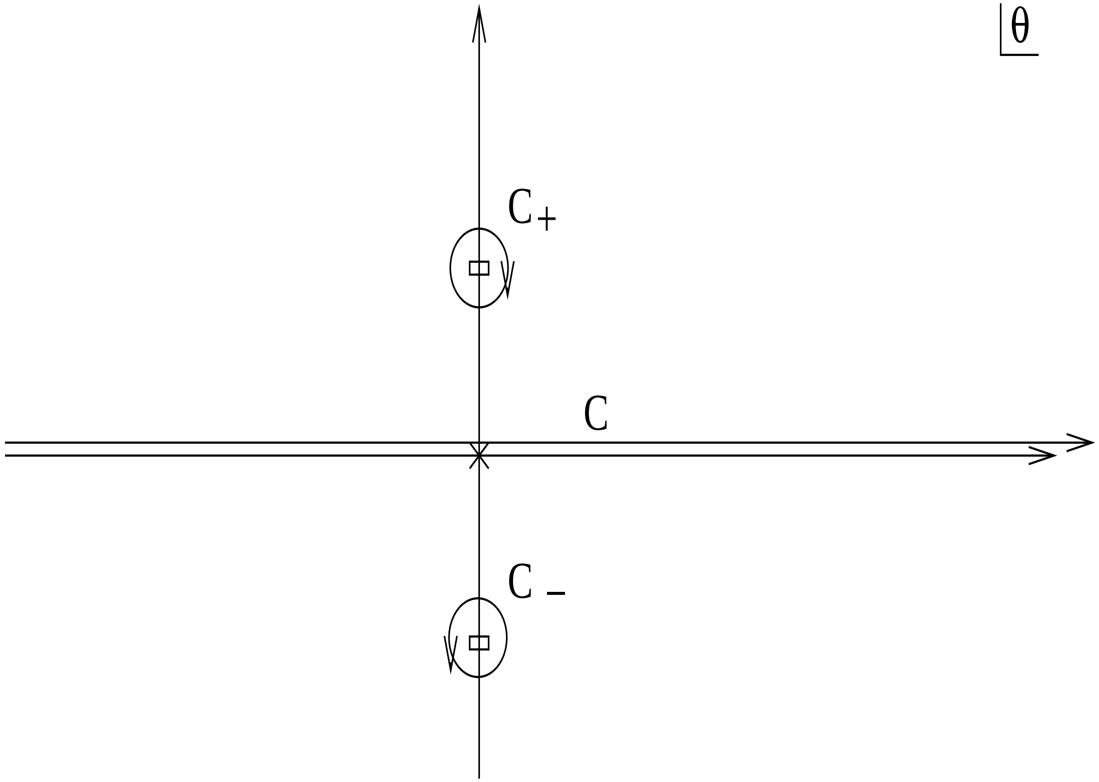
To cure this, use the idea of singularity crossing.

Singularity of $L(\theta)$: $\quad \bar{K}_\beta(iu)K_\alpha(iu)e^{-\epsilon(iu)} + 1 = 0$



Non-symmetric case (12)

Deform the integration contour and analytically continue BTBA



Non-symmetric case (13)

New form of BTBA

$$E^0_{-\alpha\beta}(R) = m\,\sin u - m\int_{-\infty}^{\infty}\frac{d\theta}{4\pi}\,\cosh\theta\,\log\left(1+\lambda_{\alpha\beta}(\theta)\,e^{-\epsilon(\theta)}\right)$$

$$\epsilon(\theta) = 2mR\cosh\theta + \log\frac{S(\theta-iu)}{S(\theta+iu)} - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\varphi(\theta-\theta')\log\left(1+\lambda_{\alpha\beta}(\theta')\,e^{-\epsilon(\theta')}\right)$$

$$1 = -\bar{K}_\beta(iu)K_\alpha(iu)e^{-\epsilon(iu)}$$

Non-symmetric case (14)

Large distance behavior: $mR \gg 1$

- $\epsilon \cong 2mR\cosh\theta$
- $\sin u \cong u = g_\alpha g_\beta \frac{e^{-mR}}{2}$
- energy difference: $\Delta E^0_{\alpha\beta} \cong m\sin u$
- crossed channel rapidity: $\kappa = (\frac{\pi}{2} - u)$

$$\Delta E^0_{\alpha\beta} \cong m\sin u = m\cosh(i\kappa)$$

Non-symmetric case (15)

- zero position

$$\bar{K}_\beta(iu)K_\alpha(iu)e^{-\epsilon(iu)} = -1$$

- cross channel:

use $K(\theta) = S(2\theta)K(-\theta)$ (crossing symmetry), $S(2i\kappa) \cong -1$
and $\epsilon(iu) \cong 2mR\cosh(iu) = -2imR\sinh i\kappa$

$$R_\beta(i\kappa)R_\alpha(i\kappa)e^{2imR\sinh(i\kappa)} = 1$$

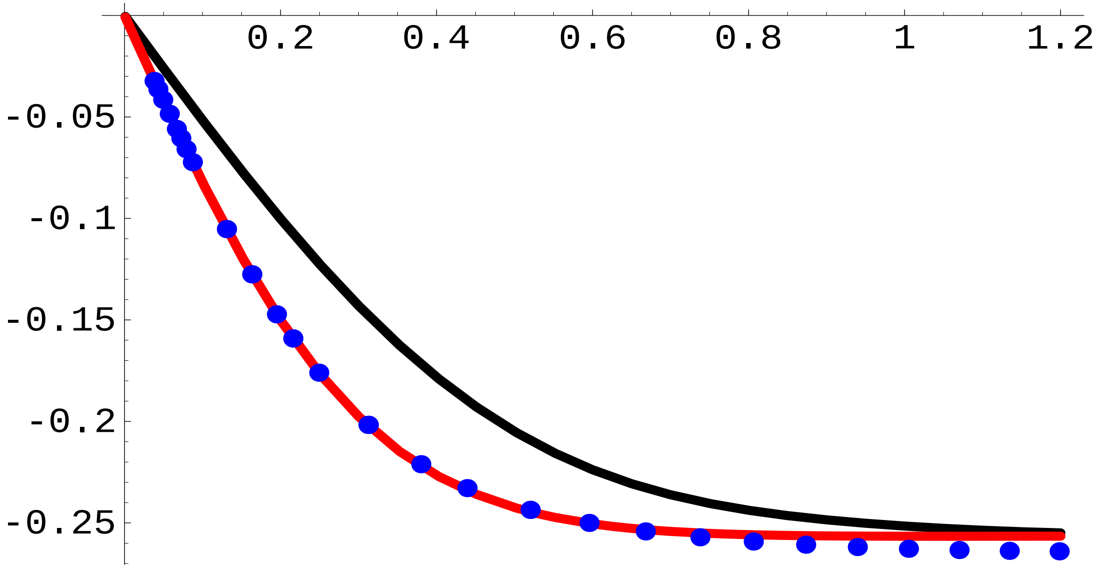
- a particle is trapped inside the strip, reflecting with

momentum $p = i\kappa \cong i\pi/2$ and
energy $\Delta E^0_{\alpha\beta} \cong m\cosh(i\kappa)$

Non-symmetric case (16)

Modified TBA with

$$b^2 = 0.8086\,, \quad \eta_a = \eta_b = \vartheta_a = \vartheta_b = i\pi/3$$

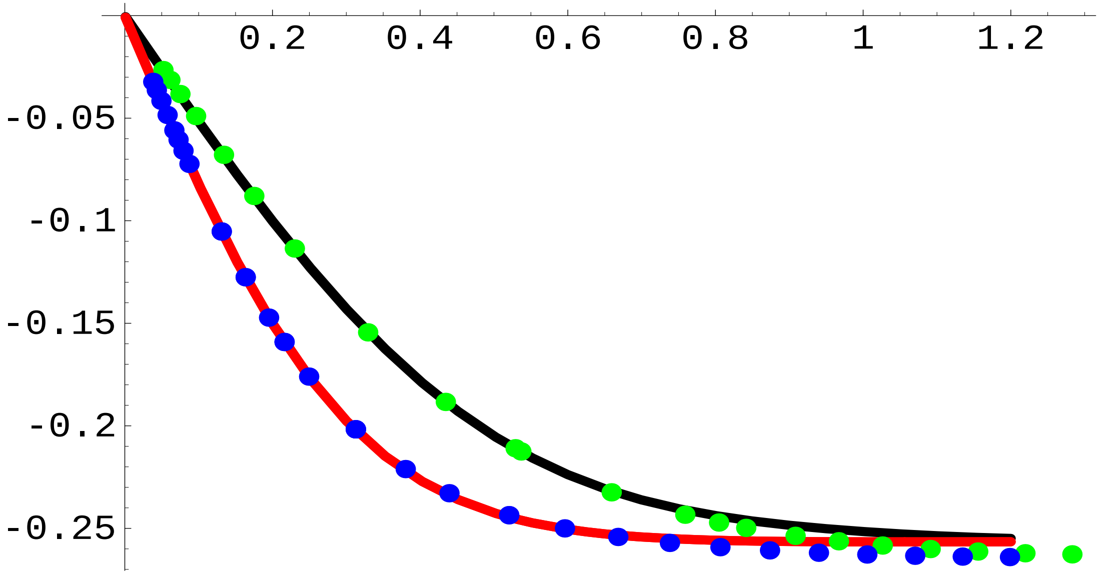


$$s_a^+ = s_b^+ = \frac{b}{\pi}(\eta_a + \vartheta_a) = i\frac{2b}{3}\,, \; s_a^- = s_b^- = \frac{b}{\pi}(\eta_a - \vartheta_a) = 0$$

$$\left(\frac{\Delta_{\rm B}(P|s_a^+,s_b^+)+\Delta_{\rm B}(P|s_a^-,s_b^-)}{2}\right), \quad \left(\frac{\Delta(P|s_a^+,s_b^-)+\Delta(P|s_a^-,s_b^+)}{2}\right)$$

Non-symmetric case (17)

Summary with $b^2 = 0.8086\,,\quad \eta_a = \eta_b = \vartheta_a = \vartheta_b = i\pi/3$



$$s_a^+ = s_b^+ = \frac{b}{\pi}(\eta_a + \vartheta_a) = i\frac{2b}{3}\,,\; s_a^- = s_b^- = \frac{b}{\pi}(\eta_a - \vartheta_a) = 0$$

$$\left(\frac{\Delta_{\rm B}(P|s_a^+,s_b^+)+\Delta_{\rm B}(P|s_a^-,s_b^-)}{2}\right),\quad \left(\frac{\Delta(P|s_a^+,s_b^-)+\Delta(P|s_a^-,s_b^+)}{2}\right)$$

Conclusion

- We have seen some non-trivial results for boundary sinh-Gordon theory, non-compact CFT perturbation theory.
- Boundary negative potential may result in non-trivial vacuum which have $c_{\text{eff}} > 1$.
  This result may lead to give some hints on boundary Toda theory with negative coefficient.
- Opposite sign of parameters (ϑ or ϕ_0) has the singularity crossing effect, which is seen already in other models.