The Abelian sandpile model: a lattice realization of a logarithmic CFT

Philippe Ruelle

Melbourne, January 2007

Forword

Historically, sandpile models have been proposed by Bak, Tang & Wiesenfeld ('87) as prototypes of self-organized critical models (SOC).

Idea was: many critical behaviours (power laws) in nature, but unlikely to result from fine-tuning $\longrightarrow$ it is the dynamics that drives the system to a critical state, even if the system is prepared in a non-critical state.

Example (BTW) $=$ sandpile, with slow addition of sand (pile builds up, then avalanches of all sizes).

Status has changes: implicit fine-tuning in the dynamics ...

[Deepak Dhar, Theoretical studies of self-organized criticality, Physica A 369 (2006) 29-70]

Important for us:

1. interesting non-equilibrium system, with stationary measure
2. lattice realization of logarithmic CFT (light on subtleties)

Plan

1. The Abelian sandpile model
   definition of 2+1 – invariant measure – Abelian property – recurrent configs – spanning trees – $c = -2$? – other boundary conditions
2. Logarithmic CFT
   why logarithms ? – inhomogeneous conformal transfomations/Ward – typical example of $c = -2$
3. Isolated dissipation
   first logarithmic field ! – non-local effect vs “local” field
4. Boundary conditions
   finite-size corrections – change of b.c. $(h = 0, -\frac{1}{8}, 0, \frac{3}{8}, 1)$ – checks
5. Height variables
   primary and logarithmic fields – with change of b.c. – predictions in bulk
6. Conclusions

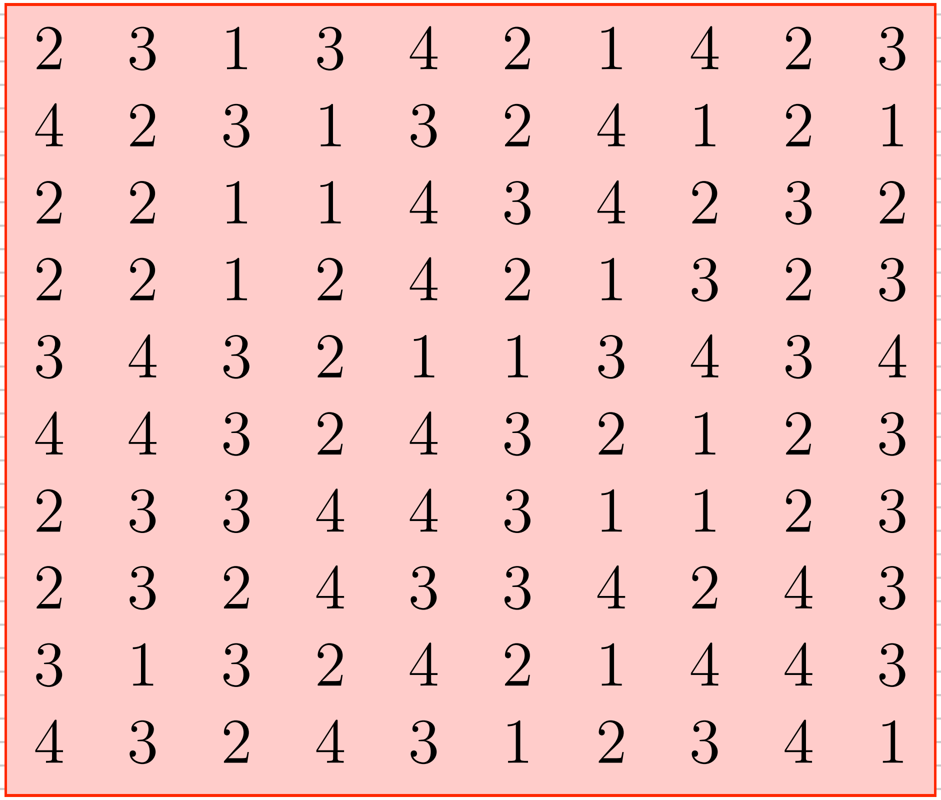
– Part I –

The Abelian sandpile model

The model

Take a grid with N sites

Attach a random variable $h_i = 1, 2, 3, 4$ to every site (h_i is # grains)



2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	2	4	3	1	2	3	4	1

configs $= 4^N$

Dynamics

The sandpile model is a stochastic dynamical system in discrete $2+1$.

Dynamics takes $\mathcal{C}_t$ into $\mathcal{C}_{t+1}$ in two steps:

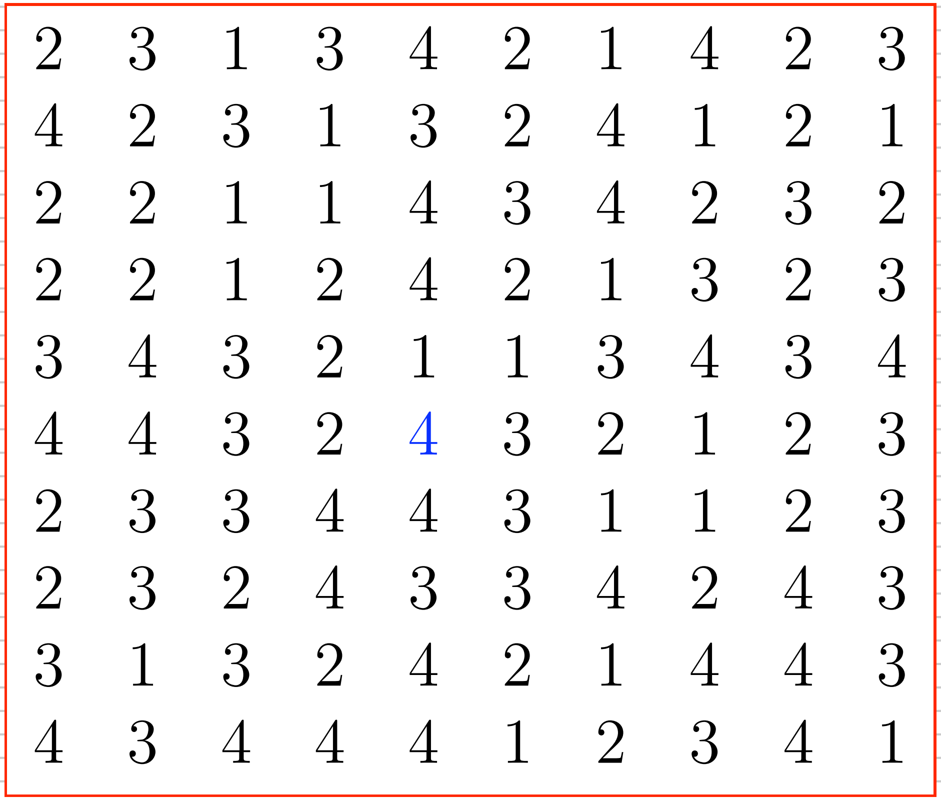
1. on random site i, drop one grain: $h_i \to h_i + 1$
2. relaxation: all unstable sites topple (avalanche)

If $h_i \geq 5$, then $\begin{cases} h_i \to h_i - 4 \\ h_j \to h_j + 1, \quad j = \text{nearest neighbour of } i \end{cases}$

Until all sites are stable again OK BECAUSE DISSIPATION !!
Resulting configuration is $\mathcal{C}_{t+1}$.

Potential chain reaction: one grain dropped can trigger a large avalanche.
System spanning avalanches will happen, and induce correlations of heights over long distances $\longrightarrow$ critical state

Typical avalanche



2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

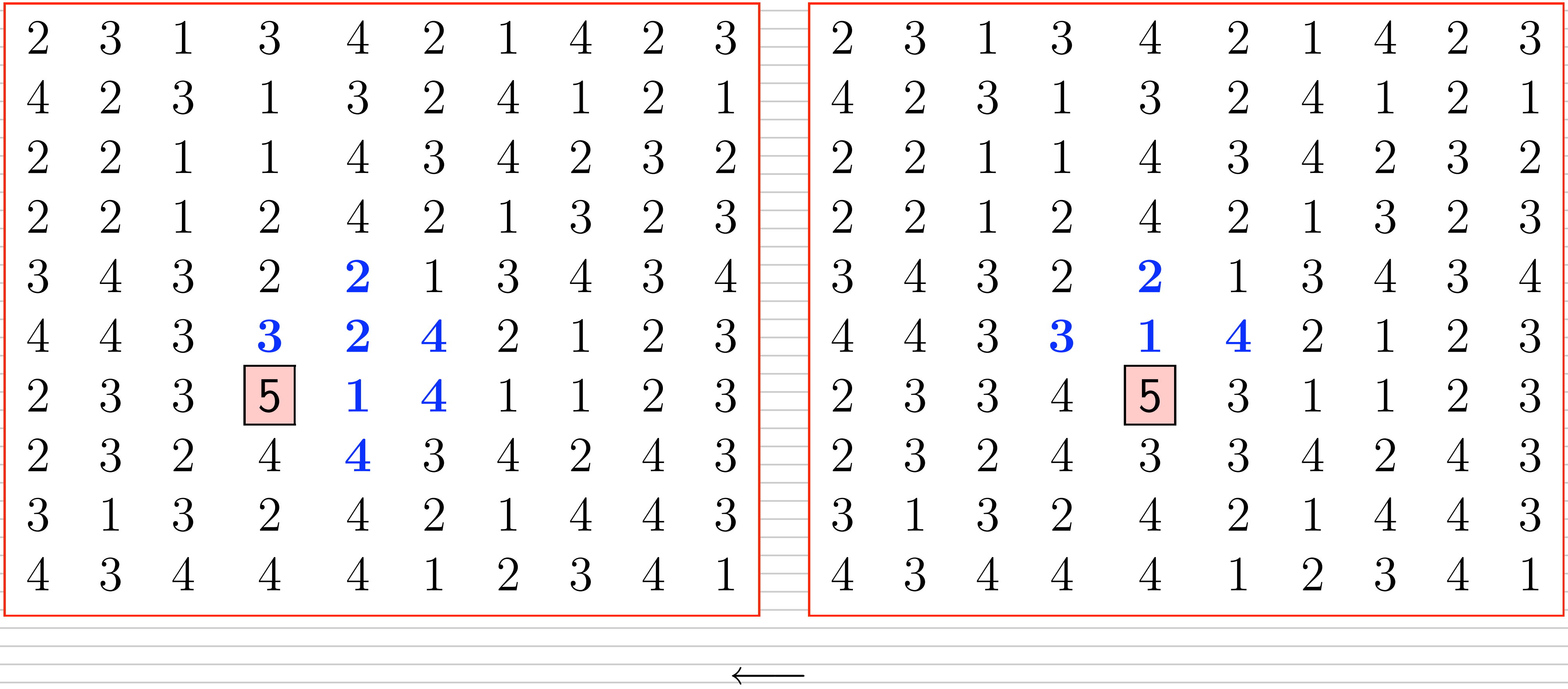
Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	$\boxed{5}$	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**1**	**4**	2	1	2	3
2	3	3	4	$\boxed{5}$	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longrightarrow$

Typical avalanche



2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**2**	**4**	2	1	2	3
2	3	3	$\boxed{5}$	**1**	**4**	1	1	2	3
2	3	2	4	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**1**	**4**	2	1	2	3
2	3	3	4	$\boxed{5}$	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

⟵

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**2**	**4**	2	1	2	3
2	3	3	$\boxed{5}$	**1**	**4**	1	1	2	3
2	3	2	4	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longrightarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**1**	**2**	**4**	1	1	2	3
2	3	2	$\boxed{5}$	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**2**	**4**	1	1	2	3
2	3	**3**	**1**	$\boxed{5}$	3	4	2	4	3
3	1	3	**3**	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**1**	**2**	**4**	1	1	2	3
2	3	2	$\boxed{5}$	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longleftarrow$

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**2**	**4**	1	1	2	3
2	3	**3**	**1**	$\boxed{5}$	3	4	2	4	3
3	1	3	**3**	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**1**	**4**	4	2	4	3
3	1	3	**3**	$\boxed{5}$	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longrightarrow$

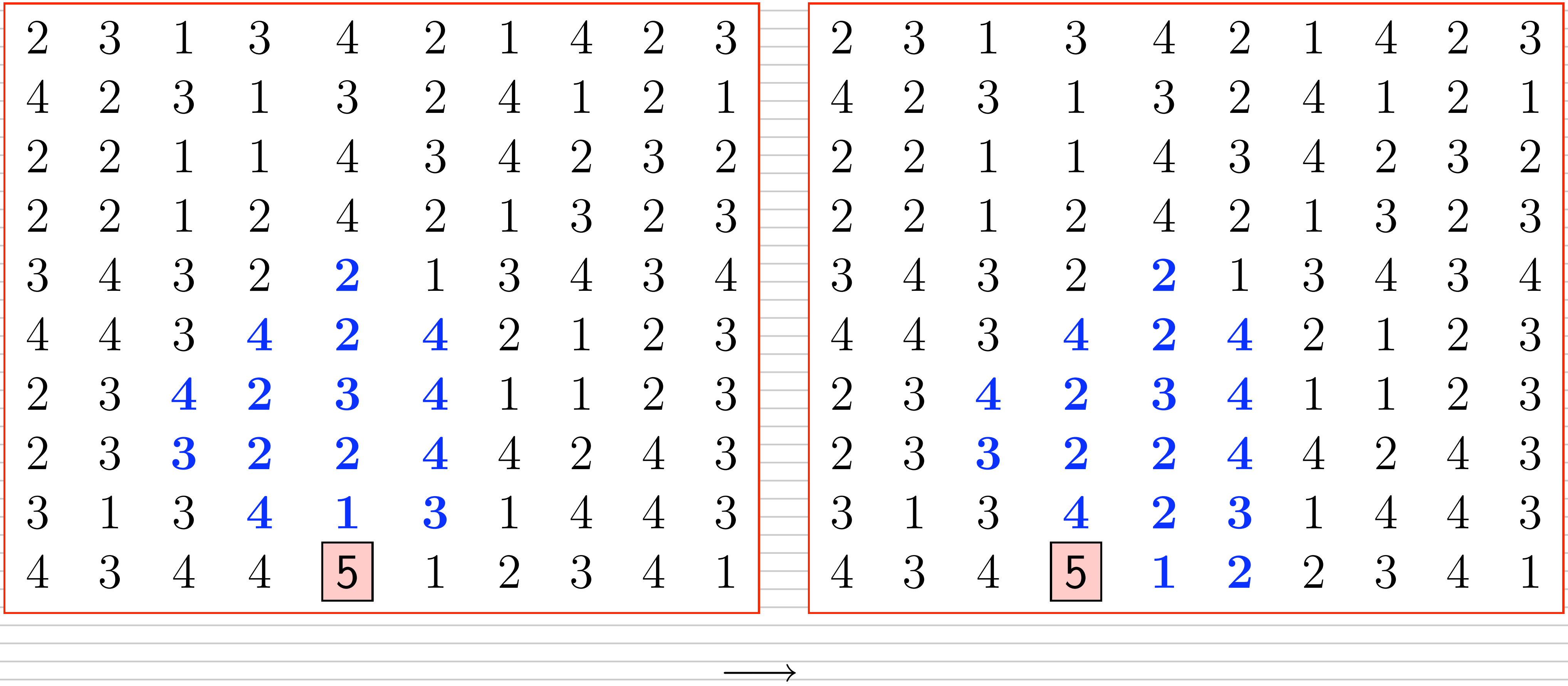
Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**1**	**3**	1	4	4	3
4	3	4	4	$\boxed{5}$	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**1**	**4**	4	2	4	3
3	1	3	**3**	$\boxed{5}$	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

⟵

Typical avalanche



2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**1**	**3**	1	4	4	3
4	3	4	4	$\boxed{5}$	1	2	3	4	1

$\longrightarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**2**	**3**	1	4	4	3
4	3	4	$\boxed{5}$	**1**	**2**	2	3	4	1

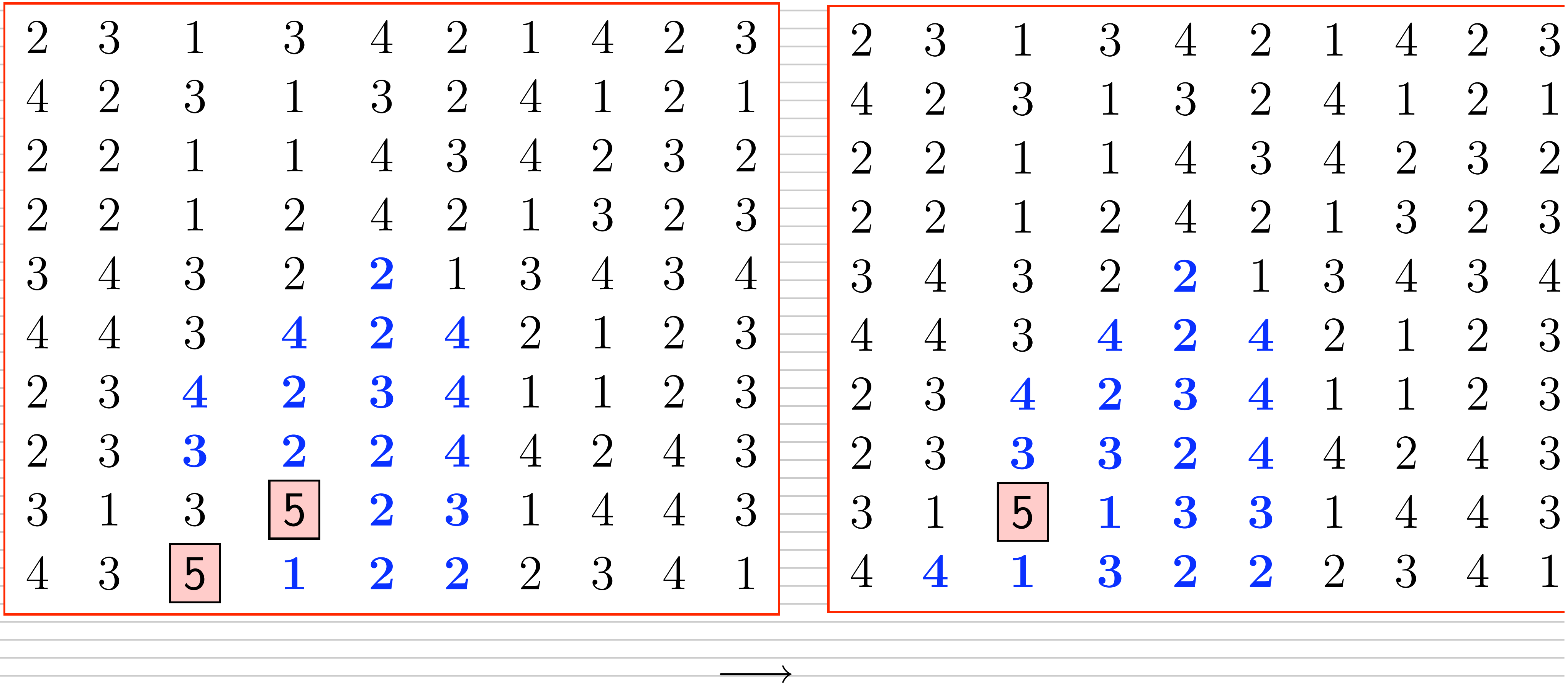
Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	$\boxed{5}$	**2**	**3**	1	4	4	3
4	3	$\boxed{5}$	**1**	**2**	**2**	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**2**	**3**	1	4	4	3
4	3	4	$\boxed{5}$	**1**	**2**	2	3	4	1

⟵

Typical avalanche



2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	5	**2**	**3**	1	4	4	3
4	3	5	**1**	**2**	**2**	2	3	4	1

⟶

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**3**	**2**	**4**	4	2	4	3
3	1	5	**1**	**3**	**3**	1	4	4	3
4	**4**	**1**	**3**	**2**	**2**	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**4**	**3**	**2**	**4**	4	2	4	3
3	**2**	**1**	**2**	**3**	**3**	1	4	4	3
4	**4**	**2**	**3**	**2**	**2**	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**3**	**2**	**4**	4	2	4	3
3	1	$\boxed{5}$	**1**	**3**	**3**	1	4	4	3
4	**4**	**1**	**3**	**2**	**2**	2	3	4	1

$\longleftarrow$

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**4**	**3**	**2**	**4**	4	2	4	3
3	**2**	**1**	**2**	**3**	**3**	1	4	4	3
4	**4**	**2**	**3**	**2**	**2**	2	3	4	1

11 topplings, 22 sites affected.

The order of topplings does not matter.

Seeding operators

Seeding operators a_i: acts on stable configurations by dropping one grain on site i and by letting the configuration relax.

Sandpile dynamics: each unit of time, a_i is applied with (uniform) probability $p_i = \frac{1}{N}$.

Because order of topplings does not matter, one can show

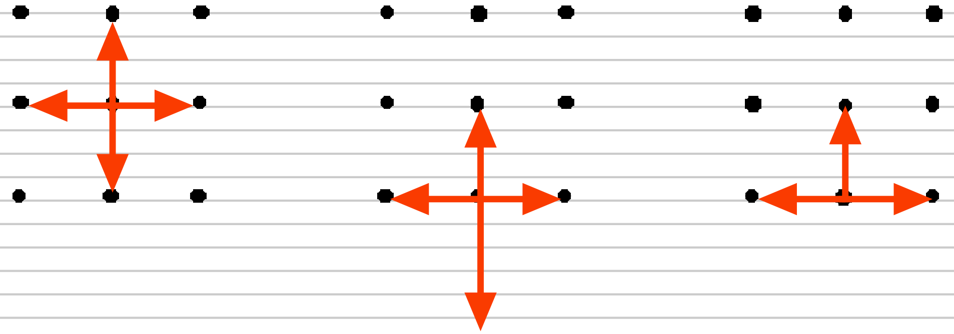
$$[a_i, a_j] = 0 \qquad \forall i, j$$

(Essentially, because toppling condition is ultra-local.)

They form an Abelian algebra, soon to be promoted to an Abelian group.

Laplacian

Redistribution of sand to neighbour sites:



If $h_i \geq 5$, then $\begin{cases} h_i \longrightarrow h_i - 4 \\ h_{\text{n.n.}} \longrightarrow h_{\text{n.n.}} + 1 \end{cases} \quad \Longleftrightarrow \quad h_j \to h_j - \Delta_{ij} \qquad \forall j$

Toppling matrix Δ is simply the Laplacian with open (Dirichlet) boundary conditions,

$$\Delta_{ij} = \begin{cases} 4 & \text{for } i = j \\ -1 & \text{for } \langle i, j \rangle \end{cases}$$

Bulk sites are conservative, open boundary sites are dissipative: when i topples, $\sum_j \Delta_{ij}$ grains leave the system, or "transferred to the sink".

Master equation

Dynamics is stochastic because seeding of sand is random.

If $P_t(\mathcal{C})$ is probability distribution at time t, then (Markov chain)

$$P_{t+1}(\mathcal{C}) = \sum_{i \in \Lambda} p_i \sum_{\mathcal{C}'} \delta(\mathcal{C} - a_i \mathcal{C}')\, P_t(\mathcal{C}')$$

The a_i are not invertible on the stable configurations: $\mathcal{C}_{\min} = \{h_i = 1\}_i$ is not in the image of the seeding operators $\Longrightarrow P_t(\mathcal{C}_{\min}) = 0$.

This is general. Configurations are either

- transient: they are not in the repeated image of the dynamics, and occur only a finite number of times ⇒ $P_t(\mathcal{C}) = 0$ for large enough t
- recurrent: they are in the repeated image of the dynamics and asymptotically occur with non-zero probability; $\exists m_i \,:\, a_i^{m_i} \mathcal{C} = \mathcal{C}$.

Invariant measure

→ time evolution flow towards recurrent configurations

→ set $\mathcal{R}$ of recurrent configurations is closed under the dynamics

→ seeding operators a_i are invertible on $\mathcal{R}$ → generate Abelian group

Behaviour of sandpile controlled by invariant measure(s) $\lim_{t\to\infty} P_t$.

We have the first important result:

The invariant measure P^*_Λ is unique and is uniform on the recurrent set $\mathcal{R}$

$$P^*_\Lambda(\mathcal{C}) = \begin{cases} \frac{1}{|\mathcal{R}|} & \text{if } \mathcal{C} \text{ is recurrent} \\ 0 & \text{if } \mathcal{C} \text{ is transient} \end{cases}$$

P^*_Λ depends on type of lattice, size of lattice, boundary conditions, number of dissipative sites, dissipation rates, ...

Recurrent set

Number of recurrent configurations ?
The group G generated by the a_i's acts irreducibly on $\mathcal{R}$: any $\mathcal{C}$ is obtained from any $\mathcal{C}'$ by a g, equivalently $\mathcal{R} = G\,\mathcal{C}^*$, for a fixed $\mathcal{C}^*$.
Therefore $|\mathcal{R}|$ is the order of G.

G is not freely generated by the a_i's, since $\prod_j a_j^{\Delta_{ij}} = 1, \forall i$.

Since G is finite Abelian, we can represent $a_j = \mathrm{e}^{2\mathrm{i}\pi\phi_j}$, such that $\sum_j \Delta_{ij}\,\phi_j = m_i$ are integers $\quad\Longrightarrow\quad \phi_j = \sum_i \Delta^{-1}_{jk}\,m_k$.

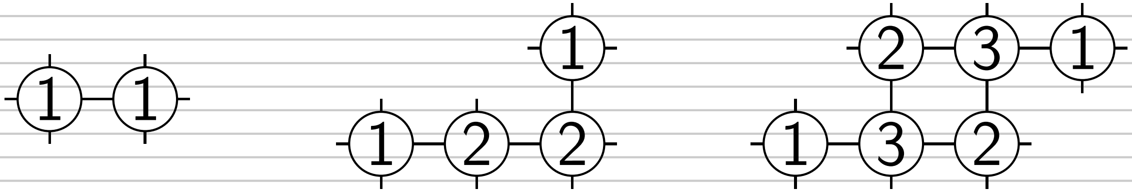
However $\{m_k\}$ and $\{m_k + \sum_l \Delta_{kl} n_l\}$ yield identical phases.

Thus distinct representations of G are labelled by integer vectors $\{m_k\}$ modulo the lattice generated by the columns $\{\Delta_{kl}\}_l$:

$$\boxed{|\mathcal{R}| = |G| = \det\Delta} \qquad (\sim 3.21^N \ll 4^N)$$

Characterization

The minimal configuration $\mathcal{C}_{\min} = \{h_i = 1\}$ is clearly not recurrent. Likewise, configurations containing the following clusters cannot be recurrent:



Forbidden Sub–Configuration: cluster F of sites s.t. every i in F has height $h_i \leq$ number of nearest neighbours in F.

A configuration is recurrent iff it has no FSCs

- Non-local characterization: requires to scan the whole configuration, and induces long range correlations of the height variables
- Makes the sandpile model a complex system: difficult to separate different length scales.

Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

4	3	1	2
2	3	2	3
1	3	2	4
2	3	4	2

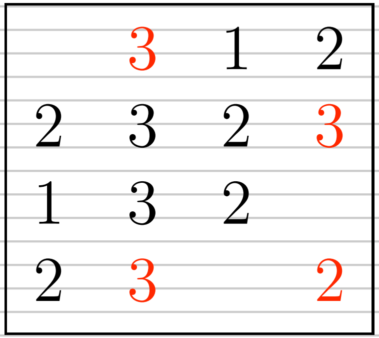
Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

	3	1	2
2	3	2	3
1	3	2	
2	3		2

Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.



	3	1	2
2	3	2	3
1	3	2	
2	3		2

Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

		1	2
2	3	2	
1	3	2	
2			

Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

		1	2
2	3	2	
1	3	2	
2			

Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

		1
2	3	2
1	3	2

Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

$$\text{the configuration } \begin{array}{|cccc|} \hline 4 & 3 & 1 & 2 \\ 2 & 3 & 2 & 3 \\ 1 & 3 & 2 & 4 \\ 2 & 3 & 4 & 2 \\ \hline \end{array} \text{ is not recurrent !}$$

Burning algorithm

To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

$$\text{the configuration } \boxed{\begin{array}{cccc} 4 & 3 & \color{red}{1} & 2 \\ \color{red}{2} & \color{red}{3} & \color{red}{2} & 3 \\ \color{red}{1} & \color{red}{3} & \color{red}{2} & 4 \\ 2 & 3 & 4 & 2 \end{array}} \text{ is not recurrent !}$$

$$\text{but the configuration } \boxed{\begin{array}{cccc} 4 & 3 & \color{red}{1} & 2 \\ \color{blue}{3} & \color{red}{3} & \color{red}{2} & 3 \\ \color{red}{1} & \color{red}{3} & \color{red}{2} & 4 \\ 2 & 3 & 4 & 2 \end{array}} \text{ is recurrent !}$$

Burning algorithm

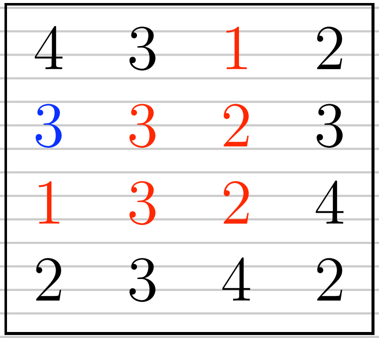
To be sure that a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

$$\text{but the configuration } \begin{array}{|cccc|}\hline 4 & 3 & 1 & 2 \\ 3 & 3 & 2 & 3 \\ 1 & 3 & 2 & 4 \\ 2 & 3 & 4 & 2 \\ \hline \end{array} \text{ is recurrent !}$$

The burning algorithm does more: keeping track of the way fire spreads in the lattice leads to spanning trees ...

Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.
This fire line defines a spanning tree.

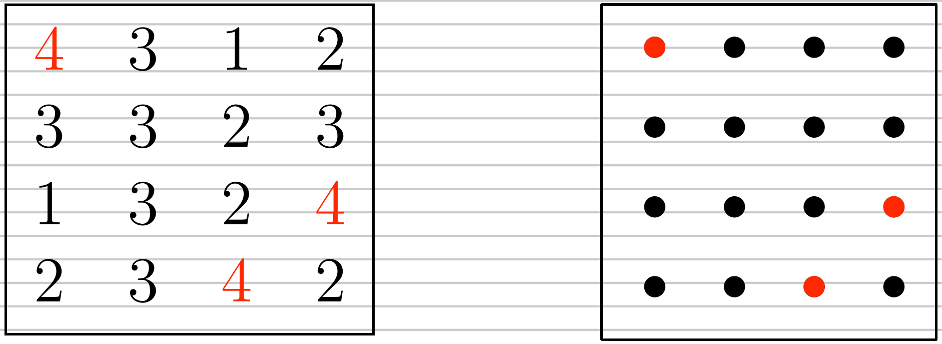


4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.

This fire line defines a spanning tree.

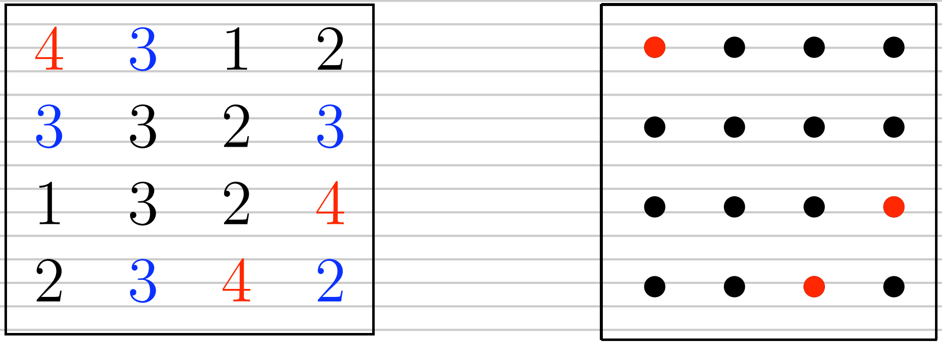


4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.

This fire line defines a spanning tree.

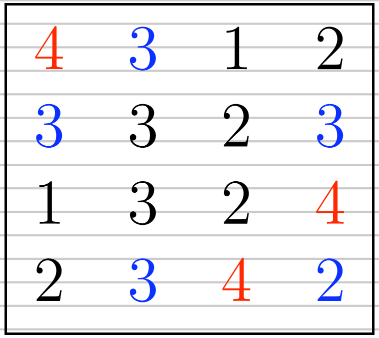


4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

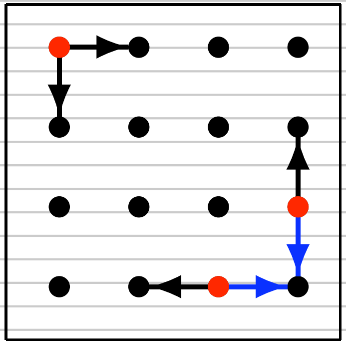
Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.

This fire line defines a spanning tree.

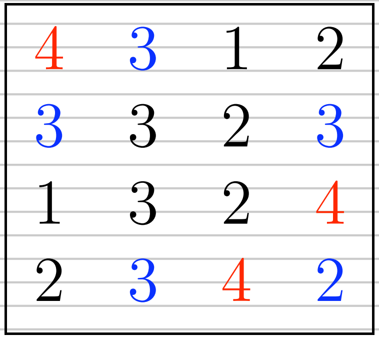


4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

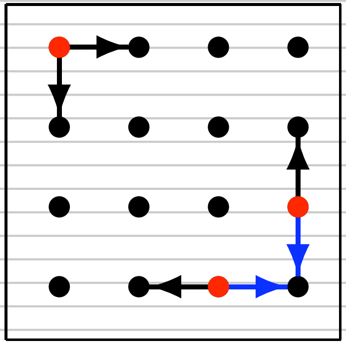


Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.
This fire line defines a spanning tree.



4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

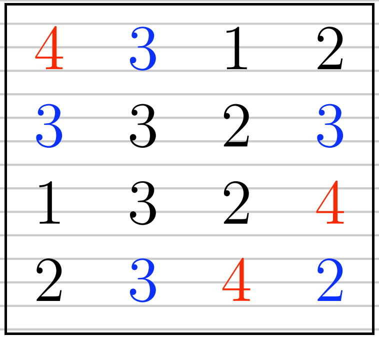


Use a prescription to select a blue arrow:

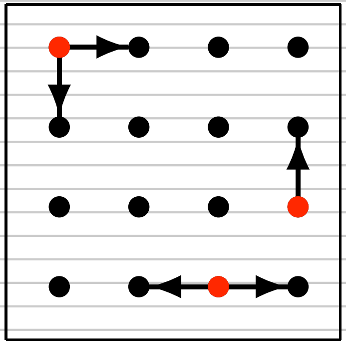
2 (height) $-$ 0 ($\#$ unburnt neigh.) $= 2 \longrightarrow$ second in $\{$N,E,S,W$\}$

Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.
This fire line defines a spanning tree.

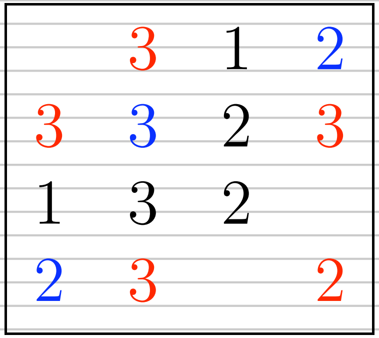


4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

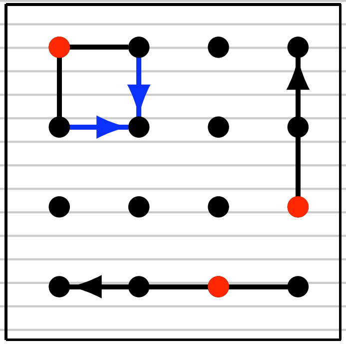


Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.
This fire line defines a spanning tree.



	3	1	2
3	3	2	3
1	3	2	
2	3		2

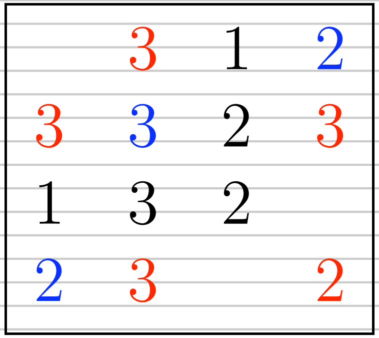


Use same prescription to select a blue arrow:

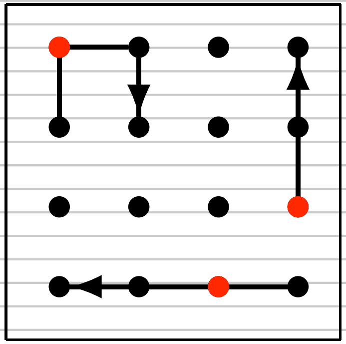
3 (height) – 2 (# unburnt neigh.) $= 1 \longrightarrow$ first in $\{$N,E,S,W$\}$

Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.
This fire line defines a spanning tree.

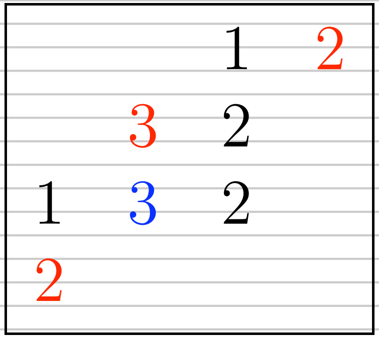


	3	1	2
3	3	2	3
1	3	2	
2	3		2

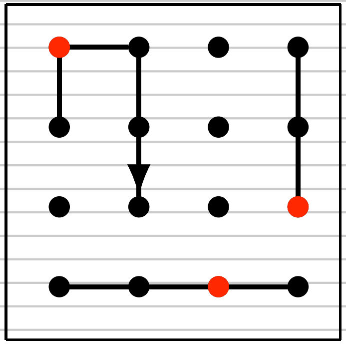


Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.
This fire line defines a spanning tree.



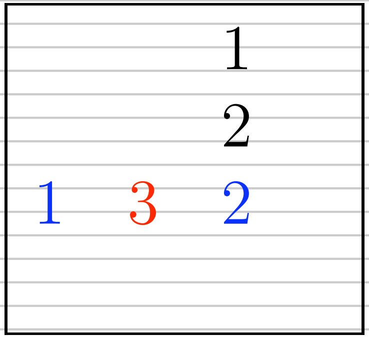
		1	2
	3	2	
1	3	2	
2			

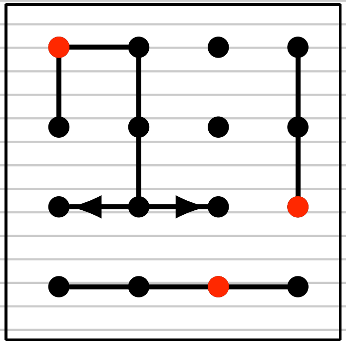


Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.

This fire line defines a spanning tree.

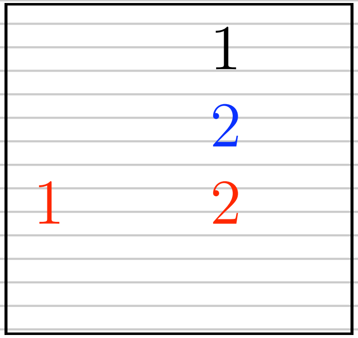


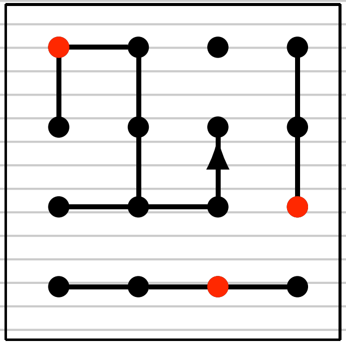


Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.

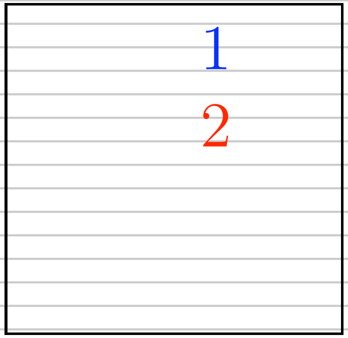
This fire line defines a spanning tree.

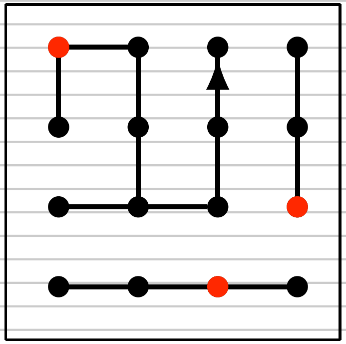




Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours were burnt the instant before ⟶ the fire propagates from neighbours to neighbours.
This fire line defines a spanning tree.

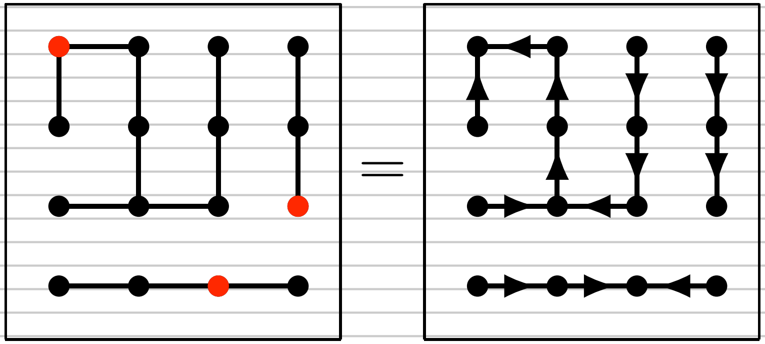




Spanning trees

This fire line defines a (disconnected) spanning tree.

4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2



Spanning tree grows from roots (red dots), which are always dissipative sites (connected to the sink).

With the prescription used, we have

recurrent configurations $\xleftrightarrow{1:1}$ spanning trees

(Kirchhoff's theorem)

ASM: summary

1. defined on a finite grid Λ, with heights $h_i = 1, 2, 3, 4$
2. necessity of dissipation (sites connected to sink)
3. configurations are either recurrent or transient
4. recurrent are in 1-to-1 correspondence with spanning trees growing from dissipative sites
5. dynamics has a unique invariant measure P^*_Λ, uniform on recurrent configurations or on spanning trees
6. non-local:

   heights are local microscopic variables but globally constrained

   $$\updownarrow$$

   spanning trees are unconstrained but global variables

Other boundary conditions

- open boundary site (dissipative)

  Under toppling, loses 4, gives 1 to three neighbours

  $$\Delta_{ii} = 4, \quad \Delta_{\langle ij \rangle} = -1, \qquad\qquad \sum_{j\in\Lambda} \Delta_{ij} > 0$$

  Height variable $1 \leq h_{\text{open}} \leq 4$.

- closed boundary site (conservative)

  Under toppling, loses 3, gives 1 to three neighbours

  $$\Delta_{ii} = 3, \quad \Delta_{\langle ij \rangle} = -1, \qquad\qquad \sum_{j\in\Lambda} \Delta_{ij} = 0$$

  Height variable $1 \leq h_{\text{closed}} \leq 3$.

  Note: all sites closed implies $\sum_j \Delta_{ij} = 0 \; \forall i \Rightarrow \det \Delta = |\mathcal{R}| = 0$.

Other b.c. (cont'd)

- boundary arrows (spanning tree variables)
  Trees are constrained to contain certain boundary bonds, with an arrow indicating the direction of the root

- periodic boundary condition
  Cylindrical geometry can be imposed provided there remain dissipation on the boundaries (torus not allowed)
- others ???

ASM: summary

1. defined on a finite grid Λ, with heights $h_i = 1, 2, 3, 4$ with prescribed boundary conditions (open, closed, arrows, ...) $\longrightarrow$ specific Δ
2. necessity of dissipation (sites connected to sink)
3. configurations are either recurrent or transient
4. recurrent are in 1-to-1 correspondence with spanning trees growing from dissipative sites
5. dynamics has a unique invariant measure P^*_Λ, uniform on recurrent configurations or on spanning trees
6. non-local:

   heights are local microscopic variables but globally constrained

   $$\updownarrow$$

   spanning trees are unconstrained but global variables

Want to show that

The thermodynamic limit $\lim_{|\Lambda|\to\infty} P_\Lambda^*$ of the invariant measure is a quantum field theoretic measure of a (logarithmic) conformal field theory

First hint at $c = -2$

Partition function measures the effective degrees of freedom

$$\boxed{Z_\Lambda = |\mathcal{R}| = \det \Delta}$$

- Finite-size correction: rectangle $L \times M$ with open b.c.

$$\lim_{M\to\infty} \frac{1}{M} \log Z_\Lambda = \frac{4G}{\pi} L + \Big(\frac{4G}{\pi} - \log(1+\sqrt{2})\Big) - \frac{\pi}{12L} + \cdots$$

First term is bulk entropy per site $\exp\frac{4G}{\pi} \simeq 3.21$

Blue term identified with $\frac{\pi c}{24L} \quad \Longrightarrow \quad \boxed{c = -2}$

- Limit $q \to 0$ of q-Potts: selects the spanning trees in FK expansion

$\longrightarrow c = 1 - \frac{6}{m(m+1)}$ with $q = 4\cos^2 \frac{\pi}{m+1} \quad \Longrightarrow \quad \boxed{c = -2}$

Questions

To confirm the relevance of conformal description, ask questions that have an answer in CFT:

1. Correlations of height variables
2. Effect of changing the boundary conditions
3. Effect of introducing additional dissipation