

– Part I –  
The Abelian  
sandpile model

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– Part II –  
Logarithmic  
conformal field  
theory

– Part III –  
Isolated dissipation  
in ASM

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– Part IV –  
Change of boundary  
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Height variables

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# – Part II – Logarithmic conformal field theory

# Rational models

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Usual features of rational models:

1. finite number of Virasoro representations
2. Vir representations are highest weight, completely reducible
3. Vir reps are completely identified by a conformal weight
4. conformal weights are bounded below
5. full, non-chiral theory basically reduces to chiral parts
6. correlation functions only have algebraic singularities
7. finite fusion (or quasi-rational)
8. chiral characters transform linearly under modular group of torus

# Log CFTs

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Typical features of logarithmic models:

1. finite number of Virasoro representations    YES/NO
2. Vir representations are highest weight, completely reducible    NO
3. Vir reps are completely identified by a conformal weight    NO
4. conformal weights are bounded below    YES
5. full, non-chiral theory basically reduces to chiral parts    NO
6. correlation functions only have algebraic singularities    NO, LOCAL ?
7. finite fusion (or quasi-rational)    YES
8. chiral characters transform linearly under modular group    NO

# Minimal models

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Minimal models are parametrized by  $(p, p')$ :

$$c = 1 - \frac{6(p - p')^2}{pp'}$$

Kac table of conformal weights

$$h_{r,s} = \frac{(p'r - ps)^2 - (p - p')^2}{4pp'}, \quad 1 \leq r < p, \quad 1 \leq s < p'$$

non-empty for  $p, p' \geq 2$ .

However the value of the central charge relevant here is

$$c = -2 \quad \longleftrightarrow \quad p = 1, p' = 2$$

# Full Kac table

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We take KT as a guiding principle

$$h_{1,s} = \frac{(s-2)^2 - 1}{8}, \quad s = 1, 2, 3, \dots$$

0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10
1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6
3	$\frac{15}{8}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3
6	$\frac{35}{8}$	3	$\frac{15}{8}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1

We observe:  $-\frac{1}{8}$  is smallest, the only negative

0 is the only one to appear twice, then  $h$  increases

$$\underline{h = -1/8}$$

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Call  $\mu$  the primary field of weight  $h_{1,2} = -\frac{1}{8}$ ,

$$\langle \mu(z)\mu(w) \rangle = A(z-w)^{1/4}.$$

The degeneracy calculation goes through:  $[2L_{-1}^2 - L_{-2}] | -\frac{1}{8} \rangle = 0$   
implies a 2nd-order ODE for the 4-pt function, where  $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$

$$\langle \mu(1)\mu(2)\mu(3)\mu(4) \rangle = (z_{12}z_{34})^{1/4}(1-x)^{1/4} [\alpha K(x) + \beta K(1-x)]$$

where  $K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-x\sin^2 t}}$  has a log singularity at  $x = 1$  ...

The log is unavoidable, either at  $x = 0$  ( $z_{12} = 0$ ) or at  $x = 1$  ( $z_{23} = 1$ ).

OPE reads

$$\mu(z)\mu(0) = \alpha z^{1/4} [\mathbb{I} + \dots] + \beta z^{1/4} [\omega(0) + \mathbb{I} \log z + \dots]$$

# Jordan block

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Second channel contains 2 fields, of weight 0 ( $= h_{1,1} = h_{1,3}$ )

$$\mu(z)\mu(0) = z^{1/4} [\omega(0) + \mathbb{I} \log z + \dots]$$

Peculiar under dilations  $z \rightarrow w = \lambda z$ ,

$$\mu(w)\mu(0) = w^{1/4} [\omega(0) - \mathbb{I} \log \lambda + \mathbb{I} \log z + \dots],$$

the field  $\omega$  picks inhomogeneous piece proportional to  $\mathbb{I}$  !

Particular case of general transformation of  $\omega$

$$\omega(w) = \omega(z) - \mathbb{I} \log \left( \frac{dw}{dz} \right).$$

Implies

$$L_0 \mathbb{I} = 0, \quad L_0 \omega = \mathbb{I} \quad \longleftrightarrow \quad L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

# Ward identities

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Two of the Möbius Ward identities becomes inhomogeneous ( $\phi = \mathbb{I}, \omega$ )

$$L_{-1} : \left( \sum_i \partial_i \right) \langle \phi(1)\phi(2) \cdots \phi(n) \rangle = 0$$

$$L_0 : \left( \sum_i z_i \partial_i + h + \hat{\delta}_i \right) \langle \phi(1)\phi(2) \cdots \phi(n) \rangle = 0$$

$$L_1 : \left( \sum_i z_i^2 \partial_i + 2z_i h + 2z_i \hat{\delta}_i \right) \langle \phi(1)\phi(2) \cdots \phi(n) \rangle = 0$$

with  $\hat{\delta} \mathbb{I} = 0$ ,  $\hat{\delta} \omega = \mathbb{I}$  (nilpotent part of  $L_0$ ).

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with  $\hat{\delta} \mathbb{I} = 0$ ,  $\hat{\delta} \omega = \mathbb{I}$  (nilpotent part of  $L_0$ ).

Immediate **consequences** on correlators:

$$\begin{aligned} \text{1-point:} \quad & \langle \mathbb{I} \rangle = 0, \quad \langle \omega(z) \rangle = a \\ \text{2-point:} \quad & \langle \omega(z)\omega(w) \rangle = -2a \log(z-w) + b \end{aligned}$$

# Why logarithms ?

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Typical when different fields have same conformal weight (not most general), due to degeneracy of the ODE for conformal blocks.

The ODE  $f'' + af' + bf = 0$  is solved by

$$f = \begin{cases} A e^{\lambda_1 x} + B e^{\lambda_2 x} & \text{if } \lambda_1 \neq \lambda_2 \\ e^{\lambda x} (A + Bx) & \text{if } \lambda = \lambda_1 = \lambda_2 \end{cases}$$

Exactly what happens in LCFT in the variable  $z = e^x$

$$f = \begin{cases} A z^{\lambda_1} + B z^{\lambda_2} & \text{if } \lambda_1 \neq \lambda_2 \\ z^{\lambda} (A + B \log z) & \text{if } \lambda = \lambda_1 = \lambda_2 \end{cases}$$

with  $\lambda_1, \lambda_2$  related to the weights of fields involved in conformal blocks.

When primaries with same weight enter a fusion, logarithms show up in correlation !

# Lagrangian realization

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Simplest and most studied LogCFT ... and precious guide !

$$S = \frac{1}{\pi} \int \partial\theta \bar{\partial}\tilde{\theta} \quad (\text{symplectic fermions})$$

- $\theta$  and  $\tilde{\theta}$  are scalar, anticommuting fields, with canonical dimension 0  
→ four fields  $\mathbb{I}$ ,  $\theta$ ,  $\tilde{\theta}$ ,  $:\theta\tilde{\theta}:$  of dimension 0.

Two bosonic fields, namely  $\mathbb{I}$  and  $\omega = : \tilde{\theta}\theta :$

- Wick contraction  $\underbrace{\theta(z, \bar{z}) \tilde{\theta}(w, \bar{w})}_{\text{contraction}} = -\log |z - w|$
- stress-energy tensor  $T(z) = -2 : \partial\theta \bar{\partial}\tilde{\theta} :$  →  $c = -2$
- identity  $\mathbb{I}$  and  $\omega = : \theta\tilde{\theta} :$  form a Jordan cell ( $\omega$  is log partner of  $\mathbb{I}$ )

$$T(z)\omega(w) = \frac{\mathbb{I}}{(z-w)^2} + \frac{\partial\omega}{z-w} + \dots$$

# Correlators

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Because of zero modes of  $\theta, \tilde{\theta}$  (remember  $\int d\theta_0 = 0$ )

$$\langle \mathbb{I} \rangle = 0, \quad \langle F(\partial\theta, \partial\tilde{\theta}) \rangle = 0.$$

However since  $\int d\theta_0 \theta_0 = 1$ , one has

$$\langle \omega(z) \rangle = \langle \tilde{\theta}\theta \rangle = 1, \quad \langle \omega F(\partial\theta, \partial\tilde{\theta}) \rangle \neq 0.$$

Most important to remember for the connection with sandpile models:

When  $\theta, \tilde{\theta}$  have zero modes (depends on surface and b.c.), all vev's which do not contain a  $\theta$  and a  $\tilde{\theta}$ , i.e. **an  $\omega$** , vanish identically !  
When they don't have zero modes, this need not be the case;  
in particular  $\langle \mathbb{I} \rangle = 1$ .

# Rational log CFT

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- Other Jordan cells, f.i.  $\phi = :\partial\theta \partial\tilde{\theta}:$  and  $\psi = :\tilde{\theta}\theta \partial\theta \partial\tilde{\theta}:$

$$\langle\phi(z)\phi(w)\rangle = 0, \quad \langle\phi(z)\psi(w)\rangle = \frac{a}{(z-w)^4}$$

$$\langle\psi(z)\psi(w)\rangle = \frac{-2a \log(z-w) + b}{(z-w)^4}$$

and more generally,  $F(\partial\theta, \partial\tilde{\theta})$  and  $\omega F(\partial\theta, \partial\tilde{\theta})$  form a logarithmic pair.

- Extended  $W$ -algebra w.r.t. to which finite number of representations

$$\text{boson : } \mathcal{V}_{-1/8}, \mathcal{R}_0, \quad \text{fermion : } \mathcal{V}_{3/8}, \mathcal{R}_1$$

Not realized in sandpiles.

This Lagrangian theory is a  $c = -2$  rational log CFT: it describes many aspects of ASM, but ... not all !!

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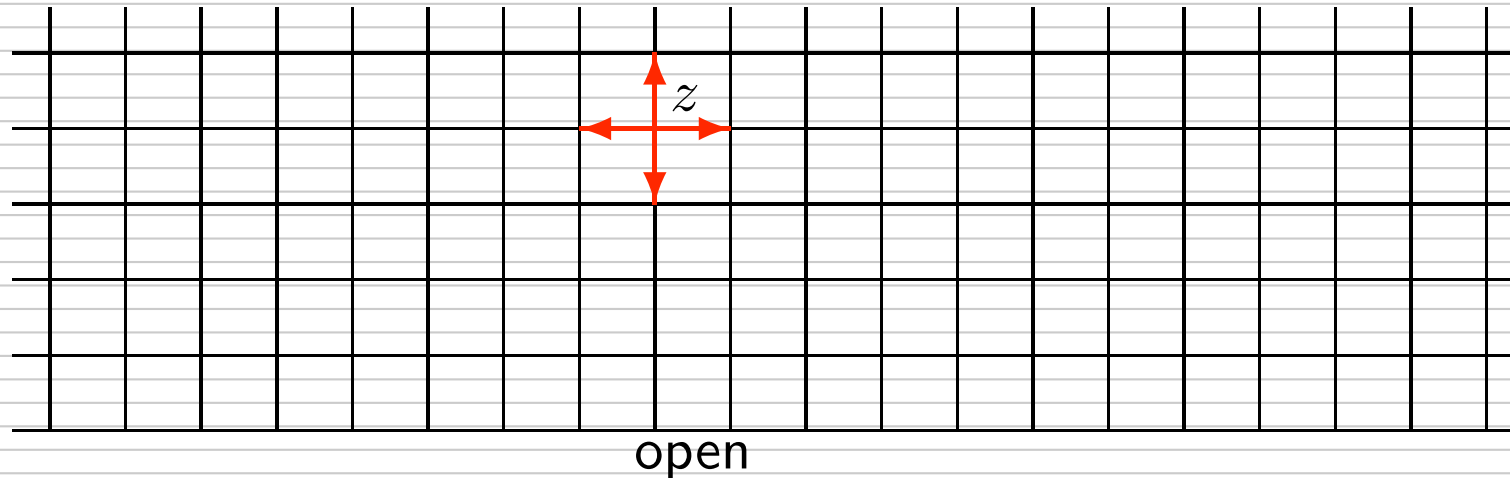
# – Part III – Isolated dissipation in ASM

## UHP: open b.c.

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So far, all sites away from boundaries are conservative.

We decide to introduce dissipation at  $z$ , in the bulk of UHP:



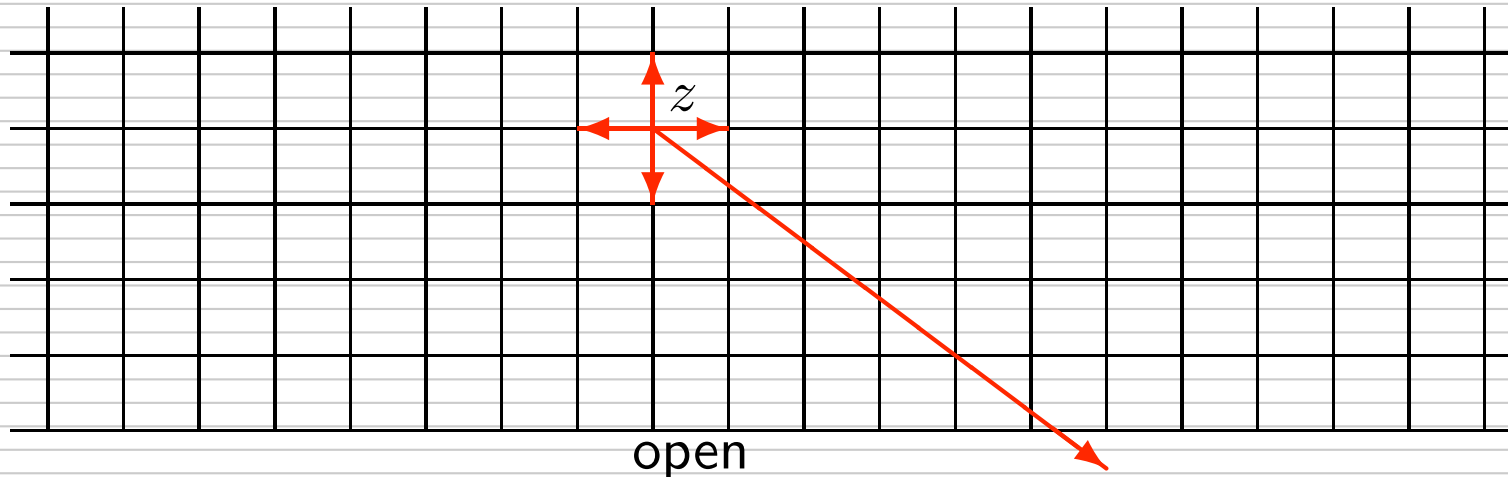
So far:  $\Delta_{ii} = 4$ ,  $\Delta_{\langle ij \rangle} = -1$  (loses 4, gives 1 to n.n.)

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So far:  $\Delta_{ii} = 4$ ,  $\Delta_{\langle ij \rangle} = -1$  (loses 4, gives 1 to n.n.)

**Minimal dissipation:**  $\Delta'_{zz} = 5$ ,  $\Delta'_{\langle zj \rangle} = -1$  (loses 5, gives 1 to n.n.)

New toppling matrix:  $\Delta'_{ij} = \Delta_{ij} + B$ ,  $B = \delta_{i,z} \delta_{j,z}$ .

## UHP: open b.c. (2)

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The effect of introducing dissipation can be measured by the fraction by which the number of recurrent configurations increases:

$$F^{\text{op}}(z) = \frac{\det \Delta'}{\det \Delta} \sim \frac{\int \phi(z) e^{-H}}{\int e^{-H}} = \langle \phi(z) \rangle = \langle \text{insertion of dissipation at } z \rangle$$

As  $\Delta' = \Delta + B$  is a local perturbation of  $\Delta$ ,

$$\frac{\det \Delta'}{\det \Delta} = \frac{\det \Delta + B}{\det \Delta} = \det[(\Delta + B)\Delta^{-1}] = \det[\mathbb{I} + B\Delta^{-1}] = 1 + G_{z,z}^{\text{op}},$$

where  $G^{\text{op}} = (\Delta^{\text{op}})^{-1}$  is the inverse Laplacian on UHP with open (Dirichlet) boundary condition.

# Green matrices

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Full plane: for  $i - j = (k, \ell)$

$$G_{i,j}^{\text{pl}} = \Delta_{i,j}^{-1} = \Delta_{(k,\ell)}^{-1} = \int_{-\pi}^{\pi} \frac{d\alpha d\beta}{8\pi^2} \frac{e^{i\alpha k + i\beta \ell}}{2 - \cos \alpha - \cos \beta}$$

All entries are infinite but differences are finite.

UHP open:

$$G_{i,j}^{\text{op}} = G_{(n_1, m_1), (n_2, m_2)}^{\text{op}} = G_{(n_1 - n_2, m_1 - m_2)}^{\text{pl}} - G_{(n_1 - n_2, m_1 + m_2)}^{\text{pl}}$$

All entries are finite.

UHP closed:

$$G_{i,j}^{\text{cl}} = G_{(n_1, m_1), (n_2, m_2)}^{\text{cl}} = G_{(n_1 - n_2, m_1 - m_2)}^{\text{pl}} + G_{(n_1 - n_2, m_1 + m_2 - 1)}^{\text{pl}}$$

All entries are infinite but differences are finite.

## UHP: open b.c. (3)

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Effect of dissipation at  $z$  is thus

$$\langle \phi(z, \bar{z}) \rangle_{\text{op}} = \frac{\det \Delta'}{\det \Delta} = 1 + G_{z,z}^{\text{op}} = 1 + G_{z,z}^{\text{pl}} - G_{z,\bar{z}}^{\text{pl}} = \frac{1}{2\pi} \log |z - \bar{z}| - \gamma_0$$

Shows that  $\phi$  is a dimension 0 field: natural candidate is  $\phi = \omega$  !

Indeed, if we think of  $\omega(z, \bar{z}) = \tilde{\theta}\theta$ , then

$$\begin{aligned} \langle \omega(z, \bar{z}) \rangle_{\text{op}} &= \langle \tilde{\theta}(z, \bar{z}) \theta(z, \bar{z}) \rangle = \lim_{w \rightarrow z} \left[ \langle \tilde{\theta}(z, \bar{z}) \theta(w, \bar{w}) \rangle - \log |z - w| \right] \\ &= \lim_{w \rightarrow z} \left[ \underbrace{\tilde{\theta}(z, \bar{z}) \theta(w, \bar{w})}_{\log |z - w|} - \log |z - w| \right] \quad (\text{open Wick}) \\ &= \lim_{w \rightarrow z} \left[ \log |z - w| - \log |z - \bar{w}| - \log |z - w| \right] \\ &= -\log |z - \bar{z}| \end{aligned}$$

Consistent to set insertion of isolated dissipation proportional to  $\tilde{\theta}\theta$  !

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## UHP: open b.c. (4)

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Insertion of dissipation at different points  $z_1, z_2, \dots, z_n$  (in bulk):

$$\Delta'_{ij} = \Delta_{ij}^{\text{op}} + B = \Delta_{ij}^{\text{op}} + \sum_k B_k, \quad B_k = \delta_{i,z_k} \delta_{j,z_k}$$

$$F^{\text{op}}(z_1 \dots z_n) = \frac{\det \Delta'}{\det \Delta^{\text{op}}} = \det[\mathbb{I} + BG^{\text{op}}] = \text{degree } n \text{ in } \log |z_k - z_l|$$

Two- and three-point correlators checked to be exactly compatible with

$$\omega(z, \bar{z}) \equiv (\text{insertion of dissipation at } z) = \frac{1}{2\pi} \theta \tilde{\theta} + \gamma_0 \mathbb{I}$$

so that

$$F^{\text{op}}(z_1 \dots z_n) = \langle \omega(z_1, \bar{z}_1) \dots \omega(z_n, \bar{z}_n) \rangle$$

computed from simple Wick contractions (no zero modes).

## UHP: closed b.c.

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Same as before but with  $\Delta^{\text{cl}}$ .

For dissipation at a single point  $z$ :

$$F^{\text{cl}}(z) = \frac{\det \Delta'}{\det \Delta^{\text{cl}}} = \frac{\det[\Delta^{\text{cl}} + B]}{\det \Delta^{\text{cl}}} = \det[\mathbb{I} + BG^{\text{cl}}] = 1 + G_{z,z}^{\text{cl}} = \infty !!$$

Normal: Denominator is 0 because there is no dissipation.

Better to normalize with respect to the situation with one dissipative site (anywhere). This regularizes the determinant ratio

$$F^{\text{cl,reg}}(z) = \frac{\det \Delta'}{\det \Delta_1^{\text{cl}}} = \frac{\det[\Delta^{\text{cl}} + B]}{\det \Delta_1^{\text{cl}}} = \frac{1 + G_{z,z}^{\text{cl}}}{1 + G_{i,i}^{\text{cl}}} = 1$$

Note: consistent with  $\langle \omega(z, \bar{z}) \rangle = 1$  as there are zero modes !!

## UHP: closed b.c. (2)

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This normalization regularizes all higher-point functions, f.i.

$$F^{\text{cl,reg}}(z_1, z_2) = \frac{\det \Delta'}{\det \Delta_1^{\text{cl}}} = \frac{\det[\Delta^{\text{cl}} + B_1 + B_2]}{\det[\Delta^{\text{cl}} + B_i]} = \frac{\det[\mathbb{I} + (B_1 + B_2)G^{\text{cl}}]}{\det[\mathbb{I} + B_i G^{\text{cl}}]}$$

All entries are infinite but have same singularity, proportional to  $2G_{0,0}^{\text{pl}}$

→ both determinants are linear in  $G_{0,0}^{\text{pl}}$ , with finite ratio.

Ratio itself is given by a minor of rank  $(n - 1)$  and therefore:

ratio has maximal degree equal to  $(n - 1)$  in  $\log |z_k - z_l|$



$(n - 1)$  Wick contractions in  $n$ -point  $\langle \omega(1) \cdots \omega(n) \rangle_{\text{cl}}$   
due to zero modes !

## UHP: closed b.c. (3)

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One finds

$$F^{\text{cl,reg}}(z_1, z_2) = \frac{1}{2\pi} \log \frac{|z_1 - z_2|^2 |z_1 - \bar{z}_2|^2}{|z_1 - \bar{z}_1| |z_2 - \bar{z}_2|} + 2\gamma_0$$

exactly equal to

$$\left\langle \left( \frac{1}{2\pi} \theta \tilde{\theta} + \gamma_0 \mathbb{I} \right) (z_1, \bar{z}_1) \left( \frac{1}{2\pi} \theta \tilde{\theta} + \gamma_0 \mathbb{I} \right) (z_2, \bar{z}_2) \right\rangle_{\text{cl}}$$

upon using

$$\langle \theta \tilde{\theta} \rangle = 2\pi, \quad \theta(z_1) \tilde{\theta}(z_2) = -\log |z_1 - z_2| - \log |z_1 - \bar{z}_2|.$$

Field identification found in open case works in closed case !

# Dissipation on boundary

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On closed boundary: essentially the same as on UHP with closed b.c.:  
 $G^{\text{cl}}$  for sites closed to the boundary have same divergence, ratios are regularized in the same way:

$$F^{\text{cl,reg}}(x) = 1 = \langle \omega_b(x) \rangle$$

$$F^{\text{cl,reg}}(x_1, x_2) = \frac{2}{\pi} \log |x_1 - x_2| + 4\gamma_0 - \frac{5}{2} = \langle \omega_b(x_1) \omega_b(x_2) \rangle$$

for  $\omega_b = \frac{1}{2\pi} \theta \tilde{\theta} + 2\gamma_0 - \frac{5}{4}$  (chiral).

On open boundary: no divergence but the insertion of dissipation on a dissipative boundary is less relevant:

$$\phi_b = \frac{2}{\pi} \partial \theta \partial \tilde{\theta} \quad (\text{chiral, dim. 2})$$

## Dissipation: summary

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The insertion of **isolated dissipation at a conservative site**

(creation of a bond to sink/root)

corresponds, in the scaling limit, to the insertion of a field  $\omega$  of weight 0,  
the logarithmic partner of the identity.