– Part IV –
Change of boundary conditions

Open versus Closed

Remember:

- open boundary site (dissipative) loses 4, gives 1 to three neighbours

$$\Delta_{ii} = 4, \quad \Delta_{\langle ij \rangle} = -1$$

  May close it (i.e. suppress dissipation) by changing diagonal element $\Delta_{ii} \to \Delta_{ii} - 1$.

- closed boundary site (conservative) loses 3, gives 1 to three neighbours

$$\Delta_{ii} = 3, \quad \Delta_{\langle ij \rangle} = -1$$

  May open it (i.e. insert dissipation) by changing diagonal element $\Delta_{ii} \to \Delta_{ii} + 1$.

Now we want to do this, not for isolated sites, but for a string of boundary sites of length n large.

Finite-size corrections

For a rectangle $L \times M$ with open b.c. on both sides

$$\lim_{M\to\infty} \frac{1}{M} \log Z_\Lambda^{\mathrm{op,op}} = \frac{4G}{\pi} L + \left(\frac{4G}{\pi} - \log(1+\sqrt{2})\right) - \frac{\pi}{12L} + \cdots$$

Calculation of determinant for open on one side and closed on the other

$$\lim_{M\to\infty} \frac{1}{M} \log Z_\Lambda^{\mathrm{op,cl}} = \frac{4G}{\pi} L + \left(\frac{2G}{\pi} - \log(1+\sqrt{2})\right) + \frac{\pi}{24L} + \cdots$$

1. boundary free energy:

$$f_{\mathrm{op}} = \frac{6G}{\pi} - \frac{1}{2}\log(1+\sqrt{2}), \qquad f_{\mathrm{cl}} = \frac{4G}{\pi} - \frac{1}{2}\log(1+\sqrt{2})$$

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1. boundary free energy: $f_{\mathrm{op}} - f_{\mathrm{cl}} = \frac{2G}{\pi}$

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1. boundary free energy: $f_{\mathrm{op}} - f_{\mathrm{cl}} = \frac{2G}{\pi}$
2. universal correction for open/closed is

$$\frac{\pi}{24L} = \frac{\pi}{24L}(c - 24h_{\min}) \qquad \longrightarrow \qquad \boxed{h_{\min} = -\tfrac{1}{8}}$$

Finite-size corrections

For a rectangle $L \times M$ with open b.c. on both sides

$$\lim_{M\to\infty} \frac{1}{M} \log Z_\Lambda^{\mathrm{op,op}} = \frac{4G}{\pi} L + \left(\frac{4G}{\pi} - \log\left(1+\sqrt{2}\right)\right) - \frac{\pi}{12L} + \cdots$$

Calculation of determinant for open on one side and closed on the other

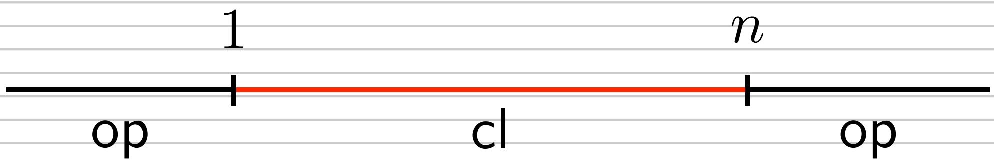
$$\lim_{M\to\infty} \frac{1}{M} \log Z_\Lambda^{\mathrm{op,cl}} = \frac{4G}{\pi} L + \left(\frac{2G}{\pi} - \log\left(1+\sqrt{2}\right)\right) + \frac{\pi}{24L} + \cdots$$

1. boundary free energy: $f_{\mathrm{op}} - f_{\mathrm{cl}} = \frac{2G}{\pi}$
2. universal correction for open/closed yields $h_{\min} = -\frac{1}{8}$

   Suggests that the b.c. changing field μ between open and closed has dimension $-\frac{1}{8}$!!

Closing the open

Consider UHP with open b.c. and close sites on interval $I = [1, n]$



Toppling matrix for original system is $\Delta^{\rm op}$, while that with closed I is

$$\Delta' = \Delta^{\rm op} - B_I\,, \qquad (B_I)_{i,j} = \delta_{i,j}\,\delta(i \in I)$$

Effect of closing I measured by fraction by which number of recurrent configurations changes

$$\frac{\det \Delta'}{\det \Delta^{\rm op}} = \frac{\det[\Delta^{\rm op} - B_I]}{\det \Delta^{\rm op}} = \det[\mathbb{I} - G^{\rm op} B_I] \quad \overset{??}{\sim} \quad \langle \mu(0)\mu(n) \rangle$$

Closing the open (2)

Ratio is

$$\frac{\det \Delta'}{\det \Delta^{\rm op}} = \det[\mathbb{I} - G^{\rm op} B_I] = \det[\delta_{i,j} - G^{\rm op}_{i,j}] = \det(s_{i-j})_{1\le i,j\le n}$$

is n-by-n Toeplitz determinant. Entries are Fourier coefficients of $\sigma(k)$

$$s_m = \int_0^{2\pi} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{-ikm} \left[\sqrt{(3-\cos k)(1-\cos k)} + \cos k - 1\right]$$

Szegö-Widom: if $\sigma(k) = (2 - 2\cos k)^\alpha\, \tau(k)$ with τ single-valued, smooth, nowhere vanishing nor divergent, then

$$\det(s_{i-j})_{1\le i,j\le n} \simeq A\, n^{\alpha^2}\, \mathrm{e}^{n t_0} + \text{expon. small}$$

for constant A and $t_0 = (\log \tau)_0$.

Closing the open (3)

Here:

$$\sigma(k) = \sqrt{(3-\cos k)(1-\cos k)} + \cos k - 1$$
$$= (2 - 2\cos k)^{1/2}\left[\sqrt{\frac{3-\cos k}{2}} - \sqrt{\frac{1-\cos k}{2}}\right]$$

Implies $\alpha = \frac{1}{2}$ and $t_0 = (\log \tau)_0 = -\frac{2G}{\pi}$. Hence

$$\frac{\det \Delta'}{\det \Delta^{\text{op}}} = A\, n^{1/4}\, \mathrm{e}^{-\frac{2G}{\pi} n} \qquad n \text{ large}$$

Exponential factor is non-universal: it is related to difference of boundary free energy for closed w.r.t. open site, $f_{\text{op}} - f_{\text{cl}} = \frac{2G}{\pi}$.

Power law matches $\langle \mu(0)\mu(n)\rangle \sim n^{-2h}$ if $h = -\frac{1}{8}$ ...

Opening the closed

Consider now the inverse situation: UHP with closed b.c. and open sites on interval $I = [1, n]$

Toppling matrix for original system is $\Delta^{\rm cl}$, while that with closed I is

$$\Delta' = \Delta^{\rm cl} + B_I\,, \qquad (B_I)_{i,j} = \delta_{i,j}\,\delta(i \in I)$$

Effect of opening I measured by fraction

$$\frac{\det \Delta'}{\det \Delta^{\rm cl}} = \frac{\det[\Delta^{\rm cl} + B_I]}{\det \Delta^{\rm cl}} = \det[\mathbb{I} + G^{\rm cl} B_I] \quad \overset{??}{\simeq} \quad \langle \mu(0)\mu(n) \rangle$$

Opening the closed (2)

Ratio is thus

$$\frac{\det \Delta'}{\det \Delta^{\rm cl}} = \det[\mathbb{I} + G^{\rm cl} B_I] = \det(s_{i-j})_{1 \le i,j \le n} = \infty$$

is again Toeplitz but infinite, for same reason as for isolated dissipation in front of closed boundary. Explicitly

$$\sigma(k) = (2 - 2\cos k)^{-1/2} \Big[\sqrt{\frac{3-\cos k}{2}} + \sqrt{\frac{1-\cos k}{2}}\Big]$$

Same remedy: compare, not with fully closed, but with situation where one dissipative site on the boundary:

$$\frac{\det \Delta'}{\det \Delta^{\rm cl}} = \det(s_{i-j})_{1 \le i,j \le n} \qquad \longrightarrow \qquad \frac{\det \Delta'}{\det \Delta_1^{\rm cl}} = \frac{1}{s_0} \det(s_{i-j})_{1 \le i,j \le n}$$

Opening the closed (3)

Regularize by using

$$\sigma_\alpha(k) = (2 - 2\cos k)^{\alpha} \left[\sqrt{\frac{3-\cos k}{2}} + \sqrt{\frac{1-\cos k}{2}}\right], \qquad \alpha > -\frac{1}{2}$$

From the analysis of singularities of numerator and denominator when $\alpha \to -\frac{1}{2}$, one gets

$$\frac{\det \Delta'}{\det \Delta_1^{\rm cl}} = A\, n^{1/4}\, \mathrm{e}^{\frac{2G}{\pi} n} \qquad n \text{ large}$$

Increasing exponential factor since $f_{\rm op} - f_{\rm cl} = \frac{2G}{\pi} > 0$.

Power law is again consistent with $h = -\frac{1}{8}$, with identical constant A:

$$\langle \mu^{\rm op,cl}(0)\, \mu^{\rm cl,op}(n) \rangle_{\rm op} = \langle \mu^{\rm cl,op}(0)\, \mu^{\rm op,cl}(n) \rangle_{\rm cl} = A\, n^{1/4} \quad \text{in scaling limit}$$

Boundary fusion

Fusion expected to contain $\mathbb{I}$: $\mu(0)\mu(n) = A\, n^{1/4}\, \mathbb{I} + \cdots$.

Closed in open: OK

$$\lim_{n\to 0} n^{-1/4} \langle \mu^{\rm op,cl}(0)\, \mu^{\rm cl,op}(n) \rangle_{\rm op} = A \,\langle\, \mathbb{I}\,\rangle_{\rm op} = A$$

Open in closed:

$$\lim_{n\to 0} n^{-1/4} \langle \mu^{\rm cl,op}(0)\, \mu^{\rm op,cl}(n) \rangle_{\rm cl} = A \,\langle\, \mathbb{I}\,\rangle_{\rm cl} = 0$$

Fusion must contain ω !

Indeed suggests two different fusions:

$$\mu^{\rm op,cl}(0)\, \mu^{\rm cl,op}(n) = A\, n^{1/4}\, \mathbb{I}^{\rm op} + \cdots$$
$$\mu^{\rm cl,op}(0)\, \mu^{\rm op,cl}(n) = A\, n^{1/4}\, [\omega(0) + \mathbb{I}^{\rm cl}\, \log n + \cdots]$$

Notice:

Consistent with the general OPE derived from 4-point function:

$$\mu(z)\mu(0) = \alpha z^{1/4}\left[\mathbb{I} + \cdots\right] + \beta z^{1/4}\left[\omega(0) + \mathbb{I}\log z + \cdots\right]$$

Two channels:

1. an ordinary representation, appropriate to an outer open b.c.
   $\longrightarrow$ the identity $\mathbb{I}$ lives on open b.c. (remember dissipation $\sim T$)
2. a logarithmic one, involving an indecomposable representation, appropriate for an outer closed b.c.
   $\longrightarrow$ the identity $\mathbb{I}$ and ω live on closed b.c.

Want to check this more deeply by actually computing 4-points ...

Closing the open ... twice

Consider UHP with open b.c. and close sites on two intervals I_1 and I_2

New toppling matrix with two closed intervals I_1 and I_2 is

$$\Delta' = \Delta^{\rm op} - B_{I_1} - B_{I_2}\,, \qquad (B_I)_{i,j} = \delta_{i,j}\,\delta(i \in I)$$

Effect of closing I_1, I_2 measured by ratio

$$\frac{\det \Delta'}{\det \Delta^{\rm op}} = \det[\mathbb{I} - G^{\rm op} B_{I_1} - G^{\rm op} B_{I_2}] \quad \stackrel{??}{\sim} \quad \langle \mu(1)\mu(2)\mu(3)\mu(4)\rangle_{\rm op}$$

No longer Toeplitz ...

Closing the open ... twice (2)

Look first on CFT side.

General solution reads ($0 \leq x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \leq 1$)

$$\langle \mu(1)\mu(2)\mu(3)\mu(4)\rangle_{\rm op} = (z_{12}z_{34})^{1/4}(1-x)^{1/4}\left[\alpha K(x) + \beta K(1-x)\right]$$

with $K(x) = \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1-x\sin^2 t}} = \begin{cases} \frac{\pi}{2} + \cdots & \text{for } x \sim 0^+ \\ -\frac{1}{2}\log\frac{1-x}{16} + \cdots & \text{for } x \sim 1^- \end{cases}$

When $z_{23} \to -\infty$ $(x \to 0)$, one should have (see previous)

$$\langle \mu(1)\mu(2)\mu(3)\mu(4)\rangle_{\rm op} \longrightarrow \langle \mu(1)\mu(2)\rangle_{\rm op}\ \langle \mu(3)\mu(4)\rangle_{\rm op} = A^2\,(z_{12}z_{34})^{1/4}$$

therefore

$$\alpha = \frac{2A^2}{\pi}, \qquad \beta = 0.$$

Closing the open ... twice (3)

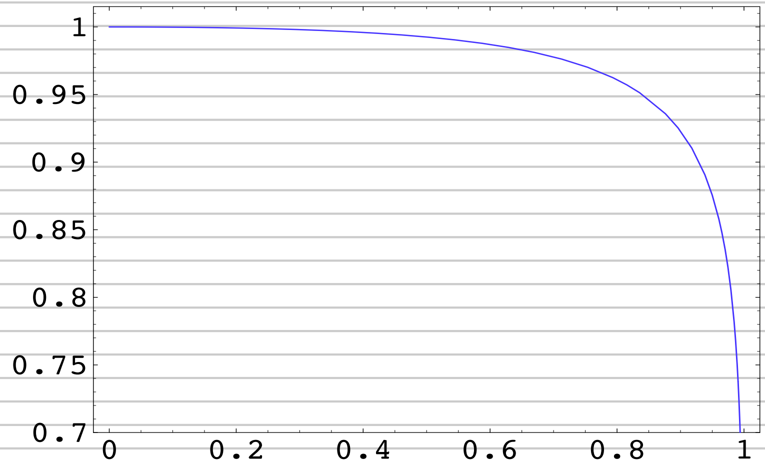
Appropriate solution is

$$\langle \mu(1)\mu(2)\mu(3)\mu(4)\rangle_{\rm op} = \frac{2A^2}{\pi}(z_{12}z_{34})^{1/4}(1-x)^{1/4}\,K(x)$$

or equivalently

$$\frac{\langle \mu(1)\mu(2)\mu(3)\mu(4)\rangle_{\rm op}}{\langle \mu(1)\mu(2)\rangle_{\rm op}\,\langle \mu(3)\mu(4)\rangle_{\rm op}} = \frac{2}{\pi}(1-x)^{1/4}\,K(x)$$

a pure function of x:



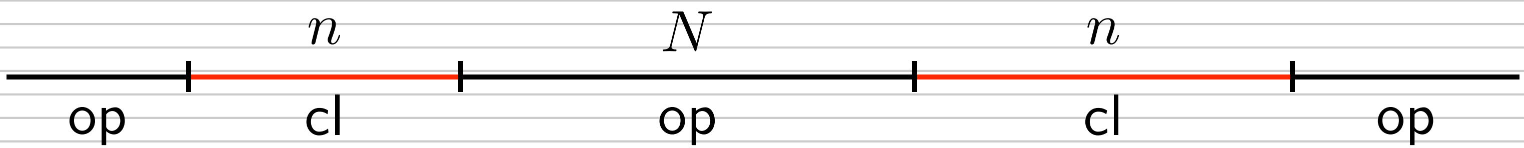
Closing the open ... twice (4)

Compare with numerical evaluation of

$$\frac{\det[\mathbb{I} - G^{\text{op}} B_{I_1} - G^{\text{op}} B_{I_2}]}{\det[\mathbb{I} - G^{\text{op}} B_{I_1}]\ \det[\mathbb{I} - G^{\text{op}} B_{I_2}]} \quad \text{as function of } x$$

(Determinant of size $|I_1| + |I_2|$) / (det of size $|I_1|$) / (det of size $|I_2|$).

We have considered



for $n = 30, 60, 90$ and $1 \leq N \leq 200$, so that

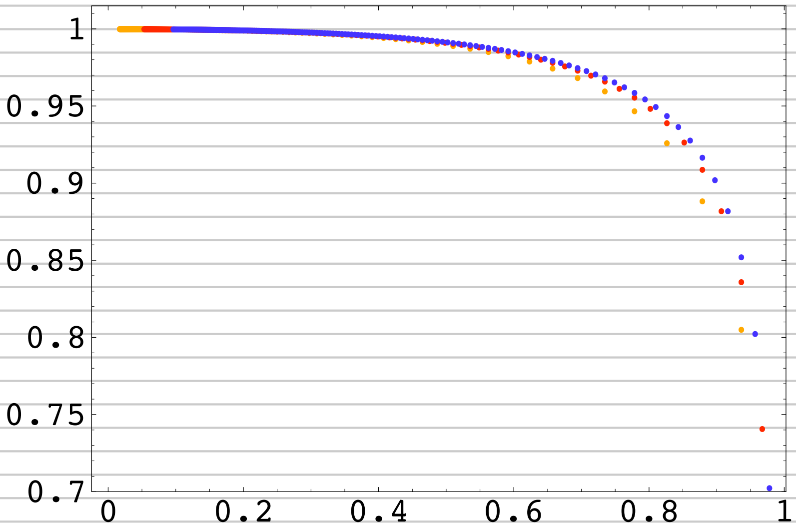
$$0.017 \leq x = \frac{z_{12} z_{34}}{z_{13} z_{24}} = \left(\frac{n}{n+N}\right)^2 \leq 1$$

Closing the open ... twice: results

For various values of n, N, the ratio

$$\frac{\det[\mathbb{I} - G^{\mathrm{op}} B_{I_1} - G^{\mathrm{op}} B_{I_2}]}{\det[\mathbb{I} - G^{\mathrm{op}} B_{I_1}] \; \det[\mathbb{I} - G^{\mathrm{op}} B_{I_2}]}$$ should be a function of x

Numerical calculations show clear data collapse



for $n = 30$ / $n = 60$ / $n = 90$: good agreement in scaling regime $x \sim 0$.

Closing the open ... twice: results

For various values of n, N, the ratio

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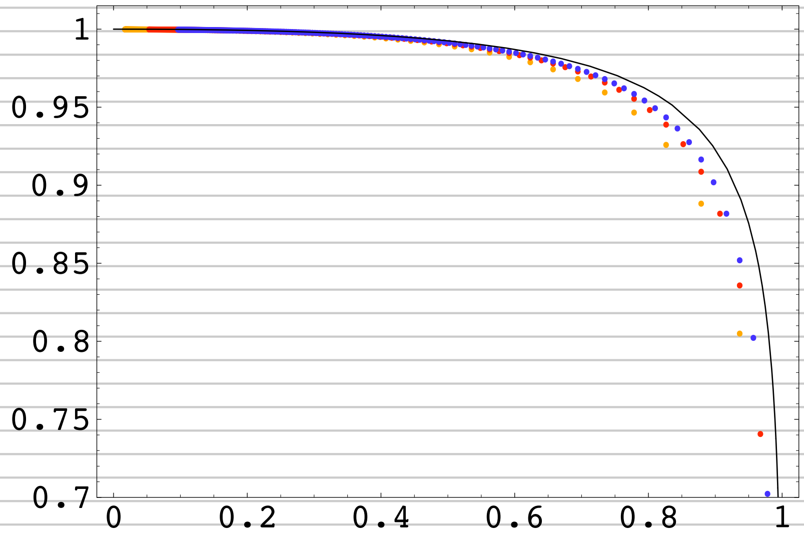


for $n = 30$ / $n = 60$ / $n = 90$: good agreement in scaling regime $x \sim 0$.

Closing the open ... twice: results

One may conclude

$$\frac{\det \Delta'}{\det \Delta^{\text{op}}} = \langle \mu^{\text{op,cl}}(1)\mu^{\text{cl,op}}(2)\mu^{\text{op,cl}}(3)\mu^{\text{cl,op}}(4)\rangle_{\text{op}}$$

$$= \frac{2A^2}{\pi}(z_{12}z_{34})^{1/4}(1-x)^{1/4}\, K(x)$$

Previous, expected fusion is fully confirmed:

- When $z_{12} \longrightarrow 0$:

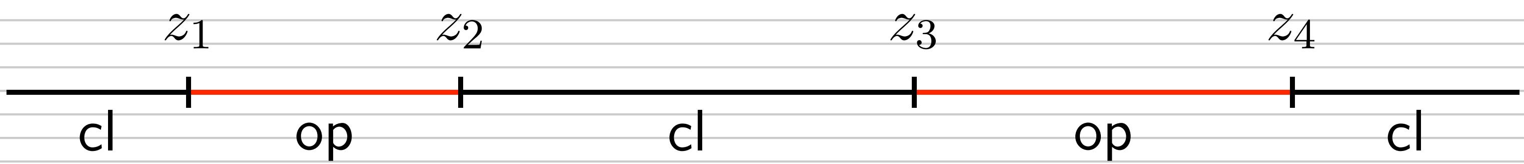
$$\langle \underbrace{\mu(1)\mu}(2)\mu(3)\mu(4)\rangle_{\text{op}} \rightarrow A^2\, z_{12}^{1/4}\, z_{34}^{1/4} + \cdots$$

- When $z_{23} \longrightarrow 0$:

$$\langle \mu(1)\underbrace{\mu(2)\mu}(3)\mu(4)\rangle_{\text{op}} \rightarrow -\frac{A^2}{\pi} z_{23}^{1/4}\left[z_{14}^{1/4}\log z_{23} + z_{14}^{1/4}\log \frac{z_{14}}{16 z_{13} z_{34}}\right] + \cdots$$

Opening the closed ... twice

Inverse situation:



Obtained from previous by cyclic permutation $(1,2,3,4) \longrightarrow (2,3,4,1)$, which changes $x \to 1-x$:

$$\frac{\det \Delta'}{\det \Delta_1^{\rm cl}} = \langle \mu^{\rm cl,op}(1)\mu^{\rm op,cl}(2)\mu^{\rm cl,op}(3)\mu^{\rm op,cl}(4)\rangle_{\rm cl}$$

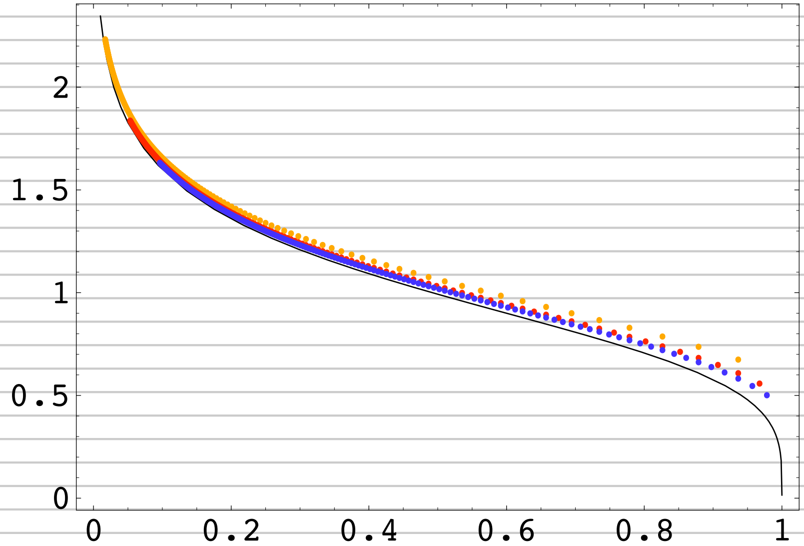
$$= \frac{2A^2}{\pi}(z_{12}z_{34})^{1/4}(1-x)^{1/4}\, K(1-x)$$

or equivalently,

$$A^{-2}(z_{12}z_{34})^{-1/4}\,\frac{\det \Delta'}{\det \Delta_1^{\rm cl}} = \frac{2}{\pi}(1-x)^{1/4}\, K(1-x)$$

Opening the closed ... twice

Satisfactory agreement



for $n = 30$ / $n = 60$ / $n = 90$ and $1 \leq N \leq 200$.

Summary: open $\leftrightarrow$ closed

The change of boundary condition from open to closed, and vice-versa, is effected, in the scaling limit, by the insertion of a chiral, boundary primary field μ with conformal dimension $-\frac{1}{8}$.

Fusion depends on outer b.c.:

$$\mu^{\rm op,cl}(0)\,\mu^{\rm cl,op}(n) = A\,n^{1/4}\,\mathbb{I}^{\rm op} + \cdots$$

$$\mu^{\rm cl,op}(0)\,\mu^{\rm op,cl}(n) = A\,n^{1/4}\,[\omega(0) + \mathbb{I}^{\rm cl}\,\log n + \cdots]$$

In second, ω is the isolated dissipation, remnant of dissipative interval.

Note: consistent with minimal fusion rules,

$$[-\frac{1}{8}] \times [-\frac{1}{8}] = (1,2) \times (1,2) = (1,1) + (1,3) = [0]_{\mathbb{I}} + [0]_{\omega}$$

Fixed arrow b.c.

Spanning trees are constrained to contain certain boundary bonds, with an arrow indicating the direction of the root

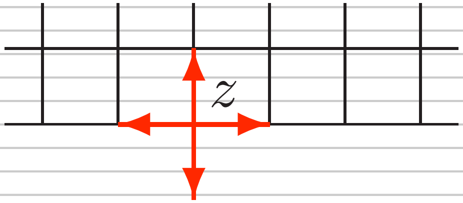
Same idea as before: insert in an open or in a closed boundary, a string of n consecutive arrows pointing to the left or to the right.

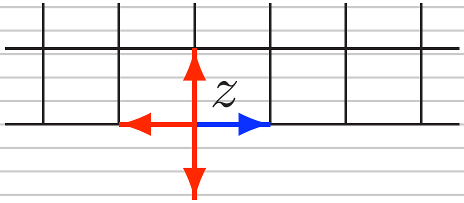
Measure the effect by the ratio:

$$\frac{\#\{\text{spanning trees with } n \text{ prescribed arrows}\}}{\#\{\text{spanning trees}\}}$$

Imposing arrows

Open boundary site





$\Delta^{\rm op}_{z,\cdot} = (\ldots, -1, 4, -1, -1, 0, \ldots)$ $\qquad$ $\Delta'_{z,\cdot} = (\ldots, -1, 4+\delta, -1, -\delta, 0, \ldots)$

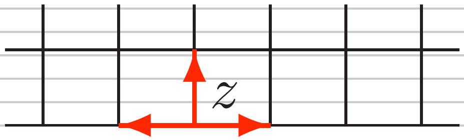
In spanning tree, only one of the four arrows is used: those with red arrow have a weight 1, those with blue arrow have a weight δ:

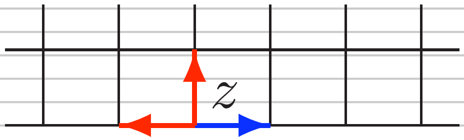
$$\lim_{\delta\to\infty} \frac{1}{\delta} \det \Delta' = \#\{\text{spanning trees with blue arrow}\}$$

$$\frac{\#\{\text{spanning trees with blue arrow}\}}{\#\{\text{spanning trees}\}} = \lim_{\delta\to\infty} \frac{1}{\delta} \frac{\det\left[\Delta^{\rm op} + \begin{pmatrix} \delta & -\delta \\ 0 & 0 \end{pmatrix}\right]}{\det \Delta^{\rm op}}$$

Imposing arrows

Same for closed boundary site





$\Delta^{\rm cl}_{z,\cdot} = (\ldots, -1, 3, -1, -1, 0, \ldots)$ $\qquad$ $\Delta'_{z,\cdot} = (\ldots, -1, 3+\delta, -1, -\delta, 0, \ldots)$

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$$\frac{\#\{\text{spanning trees with blue arrow}\}}{\#\{\text{spanning trees}\}} = \lim_{\delta\to\infty} \frac{1}{\delta} \frac{\det \left[\Delta^{\rm cl} + \begin{pmatrix} \delta & -\delta \\ 0 & 0 \end{pmatrix}\right]}{\det \Delta^{\rm cl}}$$

Inserting arrows

For n arrows inserted, must compute

$$\lim_{\delta\to\infty}\frac{1}{\delta^n}\frac{\det[\Delta+B]}{\det\Delta},\qquad B=\begin{pmatrix}\delta & -\delta & 0 & \cdots\\ 0 & \delta & -\delta & 0\\ 0 & 0 & \delta & -\delta\\ & & \cdots & \end{pmatrix}$$

Has Toeplitz form with Fisher-Hartwig singularity. Results are

open: $\displaystyle\lim_{\delta\to\infty}\frac{1}{\delta^n}\det[\mathbb{I}+G^{\mathrm{op}}B]=\mathrm{const}\times \mathrm{e}^{-\frac{4G}{\pi}n}=\langle\phi^{\mathrm{op},\rightarrow}(0)\phi^{\rightarrow,\mathrm{op}}(n)\rangle$

$\Longrightarrow$ (op,$\rightarrow$) and ($\rightarrow$,op) b.c.c.f. have weight 0

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open: $$\lim_{\delta\to\infty} \frac{1}{\delta^n} \det[\mathbb{I} + G^{\mathrm{op}} B] = \mathrm{const} \times \mathrm{e}^{-\frac{4G}{\pi} n} = \langle \phi^{\mathrm{op},\rightarrow}(0) \phi^{\rightarrow,\mathrm{op}}(n) \rangle$$

$\Longrightarrow$ (op,$\rightarrow$) and ($\rightarrow$,op) b.c.c.f. have weight 0

closed: $$\lim_{\delta\to\infty} \frac{1}{\delta^n} \det[\mathbb{I} + G^{\mathrm{cl}} B] = \mathrm{const} \times n^{-1/4} \mathrm{e}^{-\frac{2G}{\pi} n} = \langle \phi^{\mathrm{cl},\rightarrow}(0) \phi^{\rightarrow,\mathrm{cl}}(n) \rangle$$

$\Longrightarrow$ (cl,$\rightarrow$) and ($\rightarrow$,cl) b.c.c.f. have weight $-\frac{1}{8}$ and $\frac{3}{8}$

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$\Longrightarrow$ (op,$\to$) and ($\to$,op) b.c.c.f. have weight 0

closed: $$\lim_{\delta\to\infty} \frac{1}{\delta^n} \det[\mathbb{I} + G^{\mathrm{cl}} B] = \mathrm{const} \times n^{-1/4} \, \mathrm{e}^{-\frac{2G}{\pi} n} = \langle \phi^{\mathrm{cl},\to}(0) \phi^{\to,\mathrm{cl}}(n) \rangle$$

$\Longrightarrow$ (cl,$\to$) and ($\to$,cl) b.c.c.f. have weight $-\frac{1}{8}$ and $\frac{3}{8}$

interpreted as $\langle \mu(0) \nu(n) \omega(\infty) \rangle$ ...

Results

Many other checks on 3-points and 4-points lead to (in present understanding)

	open	closed	$\rightarrow$	$\leftarrow$
open	id.	$\mu = [-\frac{1}{8}]$	$\sigma = [0]$	$\sigma' = [0]$
closed	$\mu = [-\frac{1}{8}]$	id.	$\mu' = [-\frac{1}{8}]$	$\nu = [\frac{3}{8}]$
$\rightarrow$	$\sigma' = [0]$	$\nu = [\frac{3}{8}]$	id.	$\sigma'' = [0]$ (center open) $\phi = [1]$ (center closed)
$\leftarrow$	$\sigma = [0]$	$\mu' = [-\frac{1}{8}]$	$\sigma''' = [0]$	id.

All are primary fields ...