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Height variables

Introduction

Height variables: most natural but generally hardest !

Purpose = compute joint probas $P^*[h_{z_1} = a,\ h_{z_2} = b,\ \ldots]$

Plane 1-point probas computed in '91 (height 1) and in '94 (heights 2,3,4), but are completely ignored by the FT description:

$$P^*(a) = P^*[h_z = a] = \langle \delta(h_z - a) \rangle_{P^*} \neq 0 \qquad \longleftrightarrow \qquad \langle h_a(z) \rangle = 0$$

As FT describes correlation functions, the proper correspondence reads

$$\delta(h_z - a) - P^*(a) \qquad \longleftrightarrow \qquad \text{field}\ \ h_a(z)$$

under which

$$\text{scalim}\ \Big\{ P^*[h_{z_1} = a,\ h_{z_2} = b] - P^*(a)\, P^*(b) \Big\} = \langle h_a(z_1)\, h_b(z_2) \rangle$$

Introduction

The identification of scaling fields h_a requires computing lattice correlation functions of height variables ...

Easy for heights 1 (boundary or bulk)

More difficult for heights 2,3,4 on boundary (open or closed)

Sill much harder for heights 2,3,4 in bulk !

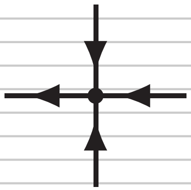
Hereafter:

1. explain how to compute and why $h = 2, 3, 4$ are hard
2. give some details for height 1
3. give results for higher heights

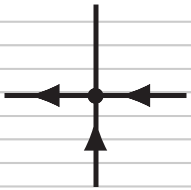
Trees, branches, leaves

Need spanning tree description of recurrent configurations of ASM.

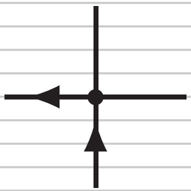
Remember the burning algorithm, building the spanning tree:



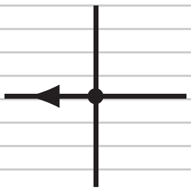
height can only be equal to 4: $P_4 = P_3 + \frac{\mathcal{N}_3}{\mathcal{N}}$



height can be equal to 3 or 4: $P_3 = P_2 + \frac{\mathcal{N}_2}{2\mathcal{N}}$



height can be equal to 2, 3 or 4: $P_2 = P_1 + \frac{\mathcal{N}_1}{3\mathcal{N}}$



height can be equal to 1, 2, 3 or 4: $P_1 = \frac{\mathcal{N}_0}{4\mathcal{N}}$

Predecessors

Previous formulae require computing the number of trees with fixed number of predecessors at given site z:

$$\mathcal{N}_k = \text{number of configs such that } z \text{ has set fire to exactly } k \text{ n.n.}$$

Huge difference between $k = 0$ and $k > 0$:

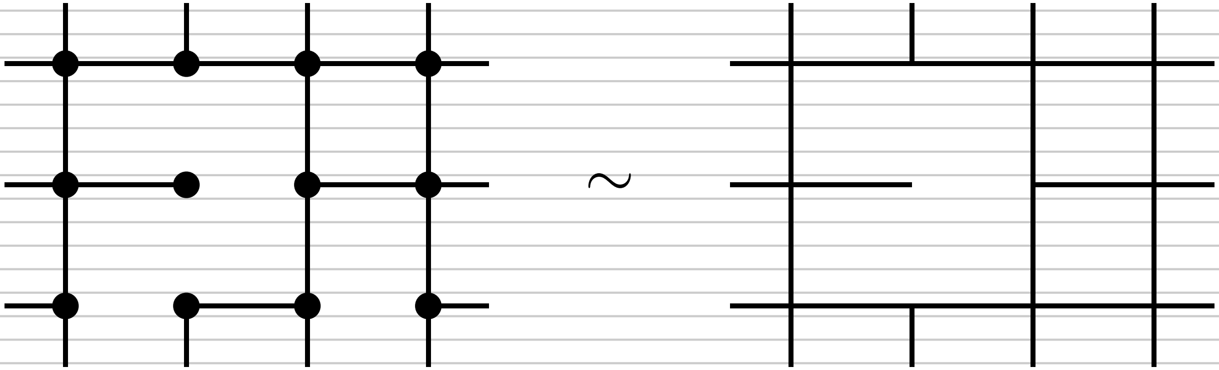
$\mathcal{N}_0$ is local: reference site is a leaf; local constraint

$\mathcal{N}_{k>0}$ is non-local: must exclude big fire path in lattice which eventually comes back to a nearest neighbour; non-local constraint

Heights 1 are easy, while heights 2, 3, 4 are hard !!

Height 1

Reference site is a leaf:



Laplacian on lattice with 3 removed bonds:

$$\Delta' = \Delta + \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathcal{N}_0 = 4 \det \Delta',$$

$$\Longrightarrow \qquad P_1(z) = \frac{\det \Delta'}{\det \Delta} = \det[\mathbb{I} + \Delta^{-1} B_z]$$ is a 4-by-4 determinant.

Height 1

1. On the **plane**: involves $G^{\rm pl}$ for sites closed to reference site z:

$$P_1 = \frac{2(\pi - 2)}{\pi^3} = 0.0736 \quad \text{(transl. inv.)}$$

2. On the **UHP with open b.c.**: $(P_1^{\rm op} = 0.104)$

$$P_1^{\rm op}(z) = P_1 + \frac{P_1}{4m^2} + \mathcal{O}(m^{-4}), \qquad m = {\rm Im}\ z$$

3. On the **UHP with closed b.c.**: $(P_1^{\rm cl} = 0.113)$

$$P_1^{\rm cl}(z) = P_1 - \frac{P_1}{4m^2} + \mathcal{O}(m^{-4})$$

Correlator $\langle h_1(z)\, h_1(\bar{z})\rangle$ of height 1 field with image

$\longrightarrow$ $h_1(z, \bar{z})$ is primary field with conformal dimensions (1,1)

Higher heights

Much harder because of non-local constraint on trees (exactly 1,2,3 predecessors).

On the **plane**, exact (integrals) expressions are known:

$$P_2 = \frac{1}{2} - \frac{1}{\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^2} - \frac{\pi - 2}{2\pi} J_2$$

with

$$J_2 = \frac{4}{\pi^2} - \frac{14}{\pi} - 8 - \frac{4\sqrt{2}}{\pi^2} \int_0^{\pi} \frac{\mathrm{d}\beta_1}{\sqrt{3 - \cos\beta_1}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\beta_2}{1 - t_1 t_2 t_3} \sin\frac{\beta_1 - \beta_2}{2}$$
$$\left[\cos\frac{\beta_1 - \beta_2}{2} - 2\cos\frac{\beta_1 + \beta_2}{2}\right] \times \left[(3 - \cos\beta_1 + \cos\beta_2)\cos\frac{\beta_1}{2} - 2\sin\beta_2 \sin\frac{\beta_1}{2}\right],$$

where $t_i = y_i - \sqrt{y_i^2 - 1}$, $y_i = 2 - \cos\beta_i$ and $\beta_3 = -(\beta_1 + \beta_2)$.

Remarkably $J_2 = 0.5 + o(10^{-12})$, but no proof !

Higher heights

Much harder because of non-local constraint on trees (exactly 1,2,3 predecessors).

On the **plane**, exact (integrals) expressions are known:

$$P_2 = \frac{1}{2} - \frac{1}{\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^2} - \frac{\pi - 2}{2\pi} J_2 = 0.1739$$

$$P_3 = \frac{1}{4} + \frac{2}{\pi} - \frac{12}{\pi^3} - \frac{8 - \pi}{4\pi} J_2 = 0.3063$$

$$P_4 = 1 - P_1 - P_2 - P_3 = 0.4461$$

Note $P_1 < P_2 < P_3 < P_4$ in agreement with forbidden subconfigurations picture.

Higher heights

On the **UHP**, even more complicated because of boundary and loss of full translation invariance. F.i.

$$\begin{aligned} P_2^{\text{open}}(m) &= \frac{\pi^3 - 3\pi^2 - 4\pi + 24}{2\pi^3} - \frac{3\pi^2 - 6\pi - 8}{8\pi^3 m} \\ &+ \frac{11\pi^2 - 40\pi + 64 + 32\pi(\pi - 2)\tilde{g}(m)}{16\pi^3 m^2} - I_{AC} - 2I_B + \dots \end{aligned}$$

$$\begin{aligned} I_B(m) &= \sum_{k\in\mathbb{Z}} \sum_{l=1-m}^{\infty} \iint_{-\pi}^{\pi} \frac{d\alpha_1 d\beta_1}{8\pi^2} \frac{e^{-il\alpha_1} - e^{i(2m+l-1)\alpha_1}}{2 - \cos\alpha_1 - \cos\beta_1} \iint_{-\pi}^{\pi} \frac{d\alpha_2 d\beta_2}{8\pi^2} \frac{e^{-il\alpha_2} - e^{i(2m+l-1)\alpha_2}}{2 - \cos\alpha_2 - \cos\beta_2} \\ &\times \iint_{-\pi}^{\pi} \frac{d\alpha_3 d\beta_3}{8\pi^2} \frac{[e^{i(1-l)\alpha_3} - e^{i(2m+l-2)\alpha_3}]e^{-i\beta_2} + [e^{-i(l+1)\alpha_3} - e^{i(2m+l)\alpha_3}]e^{i\beta_2}}{2 - \cos\alpha_3 - \cos\beta_3} \\ &\times e^{ik(\beta_1+\beta_2+\beta_3)} \det \begin{pmatrix} a' & e^{-i\beta_1} & e^{-i\beta_2} & e^{-i\beta_3} \\ b' & 1 & 1 & 1 \\ c' - d' & e^{i\alpha_1} & e^{i\alpha_2} & e^{i\alpha_3} \\ c' + d' & e^{-i\alpha_1} & e^{-i\alpha_2} & e^{-i\alpha_3} \end{pmatrix}. \end{aligned}$$

Higher heights

Asymptotic analysis for m large yields dominant contributions in SL:

$$P_i^{\mathrm{op}}(m) = P_i + \frac{1}{m^2}\left(a_i + \frac{b_i}{2} + b_i \log m\right) + \dots,$$

$$P_i^{\mathrm{cl}}(m) = P_i - \frac{1}{m^2}\left(a_i + b_i \log m\right) + \dots,$$

with explicit coefficients,

$$a_1 = \frac{\pi - 2}{2\pi^3}, \qquad b_1 = 0$$

$$a_2 = \frac{\pi - 2}{2\pi^3}\left(\gamma + \frac{5}{2}\log 2\right) - \frac{11\pi - 34}{8\pi^3}, \qquad b_2 = \frac{\pi - 2}{2\pi^3}$$

$$a_3 = \frac{8 - \pi}{4\pi^3}\left(\gamma + \frac{5}{2}\log 2\right) + \frac{2\pi^2 + 5\pi - 88}{16\pi^3}, \qquad b_3 = \frac{8 - \pi}{4\pi^3}$$

Field identifications

Strongly suggests that the height 1 and 2 fields $(h_1, h_2) \equiv (\phi, \psi)$ form a logarithmic pair of fields with conformal weights (1,1):

$$\langle \phi(z)\phi(z^*)\rangle = 0, \qquad \langle \phi(z)\psi(z^*)\rangle = \frac{a}{(z-z^*)^2}$$

$$\langle \psi(z)\psi(z^*)\rangle = \frac{1}{(z-z^*)^2}\left[a' + a\log|z-z^*|\right]$$

for $L_0\psi = \psi - \frac{1}{2}\phi$.

From

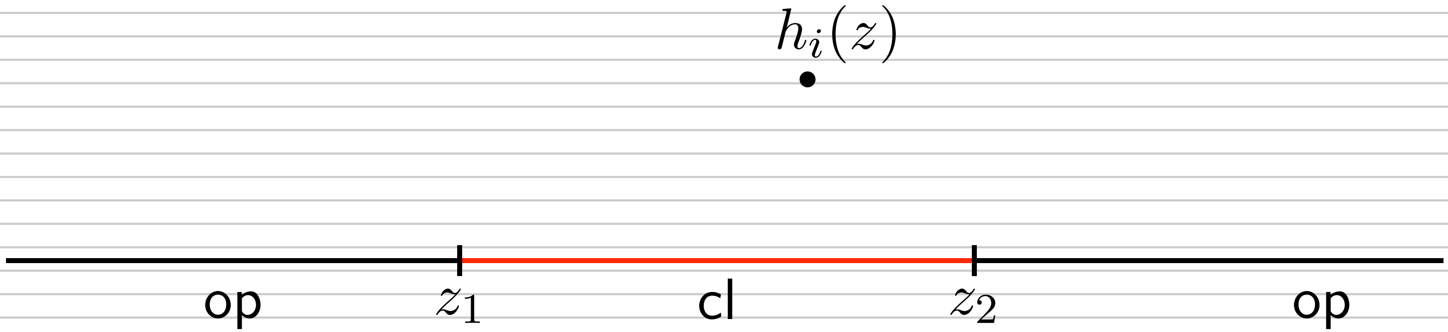
$$P_3(m) - P_3 = \alpha_3\,[P_2(m) - P_2] + \beta_3\,[P_1(m) - P_1],$$
$$P_4(m) - P_4 = \alpha_4\,[P_2(m) - P_2] + \beta_4\,[P_1(m) - P_1],$$

the height 3 and 4 fields h_3, h_4 are linear combinations of h_1 and h_2.

Checks ??

Consider two types of checks:

1. consistency within CFT: $P_i^{\rm op}(z)$ and $P_i^{\rm cl}(z)$ refer to different boundary conditions $\longrightarrow$ can be related by b.c. changing field μ

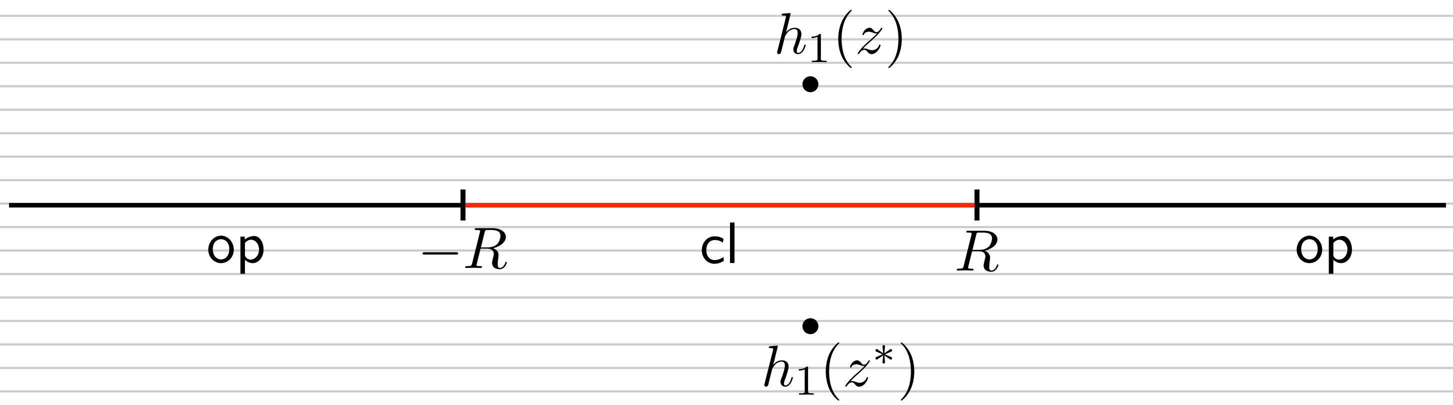


Look at limits of

$$\langle \mu(z_1)\mu(z_2)h_i(z,\bar z)\rangle \quad \text{when } z_1, z_2 \to 0 \text{ or } -z_1, z_2 \to +\infty$$

2. comparison of CFT predictions with simulation data, f.i. for finite-size corrections

Check for height 1



Fields μ (weight $-\frac{1}{8}$) and h_1 (weight 1) are degenerate at level 2:

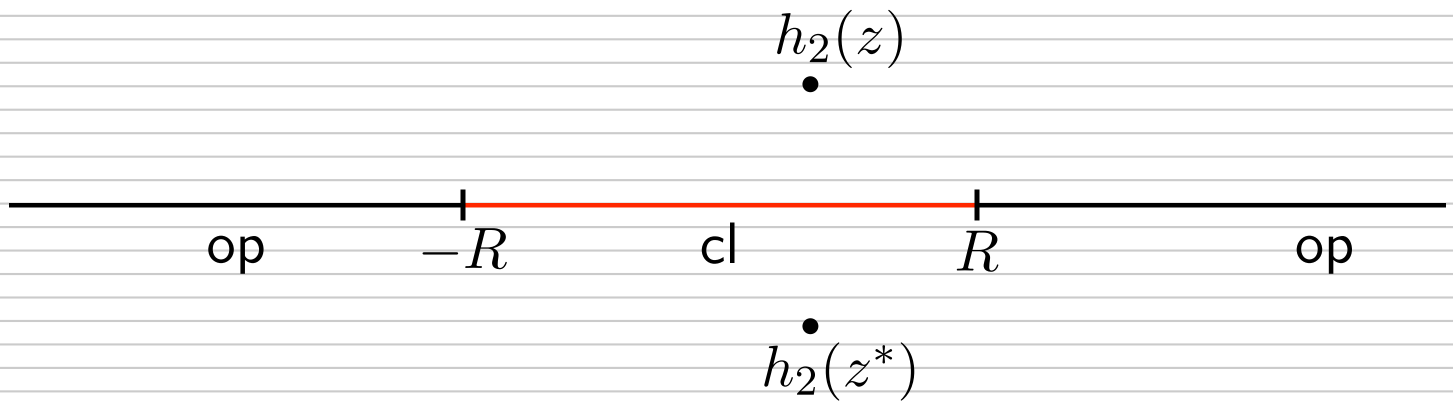
$$(2R)^{-1/4}\Big\langle \mu(-R)\mu(R)h_1(z)h_1(z^*)\Big\rangle_{\text{op}} = \frac{\alpha}{m^2}\frac{2-x}{\sqrt{1-x}}, \qquad 1-x = \frac{(R-z)(R+z^*)}{(R-z^*)(R+z)}$$

$$\lim_{R\to 0}(2R)^{-1/4}\Big\langle \mu(-R)\mu(R)\,h_1(z,z^*)\Big\rangle_{\text{op}} = \frac{2\alpha}{m^2} = \frac{P_1}{4m^2} = P_1^{\text{op}}(z) - P_1$$

$$\lim_{R\to +\infty}(2R)^{-1/4}\Big\langle \mu(-R)\mu(R)\,h_1(z,z^*)\Big\rangle_{\text{op}} = -\frac{2\alpha}{m^2} = -\frac{P_1}{4m^2} = P_1^{\text{cl}}(z) - P_1$$

(monodromy of $1-x$ around 1)

Check for height 2



More complicated: h_2 is logarithmic, and not quasi-primary:

$$T(z)h_2(w,\bar{w}) = \frac{\rho(w,\bar{w})}{(z-w)^3} + \frac{h_2(w,\bar{w}) - \frac{1}{2}h_1(w,\bar{w})}{(z-w)^2} + \frac{\partial h_2(w,\bar{w})}{z-w} + \ldots$$

$$T(z)\rho(w,\bar{w}) = \frac{\partial\rho(w,\bar{w})}{z-w} + \ldots$$

$$\bar{T}(\bar{z})\rho(w,\bar{w}) = \frac{\kappa}{(\bar{z}-\bar{w})^3} + \frac{\rho(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\rho(w,\bar{w})}{\bar{z}-\bar{w}} + \ldots$$

Check for height 2

$\Longrightarrow$ Ward identities and 2nd order ODE (from μ) are inhomogeneous: 9 correlators to compute $\langle \mu\mu\lambda\lambda \rangle$ with $\lambda = \rho, h_1, h_2$!

Imposing physical conditions (no log when $R \to 0$, regular if $R \to +\infty$),

$$(2R)^{-1/4}\Big\langle \mu(-R)\mu(R)\, h_2(z,z^*) \Big\rangle_{\rm op} = \frac{1}{2m^2}\frac{2-x}{\sqrt{1-x}}\Big\{ b\log m + a + \frac{b}{4} + \frac{b}{8}\frac{2-x}{\sqrt{1-x}} + 2b\frac{m^2}{(R+z)(R-z^*)}\Big[\frac{1}{x-2} + \frac{1}{2\sqrt{1-x}}\Big]\Big\}.$$

$$\lim_{R\to 0}(2R)^{-1/4}\Big\langle \mu(-R)\mu(R)\, h_2(z,z^*) \Big\rangle_{\rm op} = \frac{1}{m^2}[a + \frac{b}{2} + b\log m] = P_2^{\rm op}(z) - P_2$$

$$\lim_{R\to +\infty}(2R)^{-1/4}\Big\langle \mu(-R)\mu(R)\, h_2(z,z^*) \Big\rangle_{\rm op} = -\frac{1}{m^2}[a + b\log m] = P_2^{\rm cl}(z) - P_2$$

Exactly reproduces the lattice results ... (OK for heights 3, 4)

Remark

When $R \to 0$:

$$\lim_{R\to 0} (2R)^{-1/4} \Big\langle \mu(-R)\mu(R)\, h_i(z,z^*) \Big\rangle_{\rm op} = \langle h_i(z,z^*)\rangle_{\rm op}$$

as follows from $\mu(-R)\mu(R) = (2R)^{-1/4}\, \mathbb{I} + \dots$

When $R \to +\infty$:

$$\lim_{R\to +\infty} (2R)^{-1/4} \Big\langle \mu(-R)\mu(R)\, h_i(z,z^*) \Big\rangle_{\rm op} = \langle \omega(\infty) h_i(z,z^*)\rangle_{\rm cl}$$

because it does not contain the log singularity of the OPE

$$\mu(-R)\mu(R) = (2R)^{-1/4} [\omega(\infty) + \mathbb{I} \log 2R + \dots]$$

Need for dissipation ...

Infinite strip

Map UHP with closed on $\mathbb{R}_-$ and open on $\mathbb{R}_+$ onto infinite strip

by conformal map $w = \frac{L}{\pi} \log z$ for width L.

The 1-pt probability profiles are transformed:

$$P_i^{\mathrm{cl|op}}(z) - P_i = -\frac{z+z^*}{|z|(z-z^*)^2}\left\{2b_i \log\left|\frac{z-z^*}{2}\right| + 2a_i + \frac{b_i}{2} + \frac{b_i}{4}\frac{z+z^*}{|z|}\right\}$$

$$P_i^{\mathrm{strip}}(w) - P_i = \left(\frac{\pi}{L}\right)^2 \frac{\cos(\pi v/L)}{\sin^2(\pi v/L)}\left\{a_i + \frac{b_i}{4}\left[1+\cos\left(\frac{\pi v}{L}\right)\right] + b_i \log\left(\frac{L}{\pi}\sin\left(\frac{\pi v}{L}\right)\right)\right\} + \frac{b_i \pi^2}{4L^2}$$

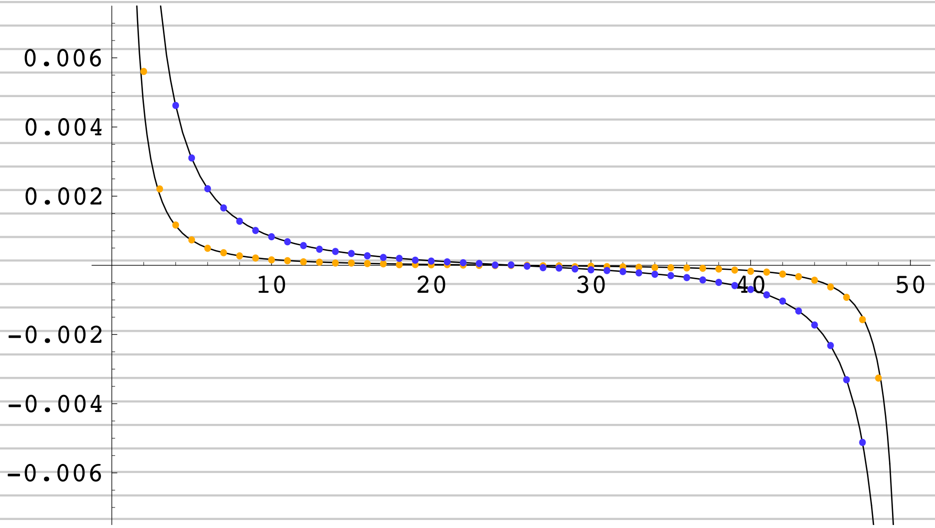
Simulations

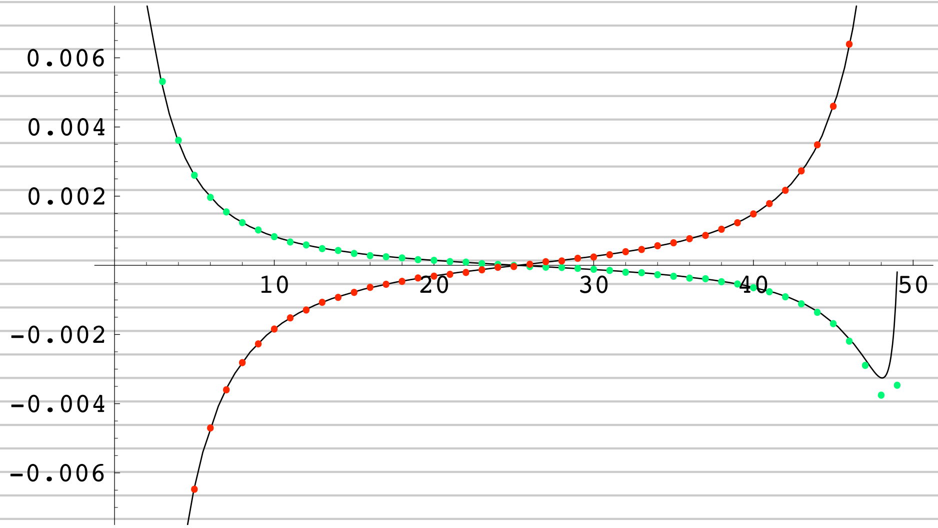
ASM dynamics on finite rectangle $M \times L$

- $M/L = 4$ to reproduce infinite strip, and $L = 50$ to approach SL
- choose open b.c. on far sides to speed up relaxation process
- run dynamics from many initial recurrent configs, and sample every 50th config to avoid correlations in data
- collect statistics of height occurrences along medial line of over 10^{10} recurrent configs (note $|\mathcal{R}| \sim 10^{5045}$ ...)
- subtract bulk probabilities P_i and compare with CFT curves

CFT vs simulations

Results for subtracted probabilities $P_i^{\rm strip}(v) - P_i$





for height 1   height 2   height 3   height 4.

Solid curves are

$$P_i^{\rm strip}(w) - P_i = \left(\frac{\pi}{L}\right)^2 \frac{\cos(\pi v/L)}{\sin^2(\pi v/L)} \left\{a_i + \frac{b_i}{4}\left[1+\cos\left(\frac{\pi v}{L}\right)\right] + b_i \log\left(\frac{L}{\pi}\sin\left(\frac{\pi v}{L}\right)\right)\right\} + \frac{b_i \pi^2}{4L^2}$$

Height fields

All tests confirm following field identifications for height variables:

The height 1 field h_1 is a primary field with weights (1,1).

The others three h_2, h_3, h_4 are all, up to normalizations, the same field, equal to the logarithmic partner of h_1.

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Good number of features well understood:

- 4 boundary conditions identified, leading to b.c. changing fields with conformal weights $0, -\frac{1}{8}, 0, \frac{3}{8}, 1$
- isolated dissipation, on boundary or in bulk, with and without change of b.c.; bulk, boundary and bulk-boundary fusions checked
- boundary height variables on closed and open boundaries (not log)
- bulk height variables properly identified (log fields), with and without change of b.c.
- fully dissipative model, no longer critical, described by massive perturbation of $c = -2$

Open issues:

- relevant LCFT likely to be non-rational: complete its identification
- look for other boundary conditions
- identify new bulk obervables
- establish relationships with other models
- cylinder and torus partition functions: new meaning ?

Perspective:

Avalanche observables