Lecture 2 alive beginning:

We end by the local YBE as a candidate for AUX problem:

Picture

$$L_{\alpha\beta}[A_1]\, L_{\alpha\gamma}[A_2]\, L_{\beta\gamma}[A_3] = L_{\beta\gamma}[A_3']\, L_{\alpha\gamma}[A_2']\, L_{\alpha\beta}[A_1']$$

where

$$A_i' = R_{123}\, A_i\, R_{123}^{-1} \xmapsto{\hbar \to 0} A_i' = f_i(A_1 A_2 A_3)$$

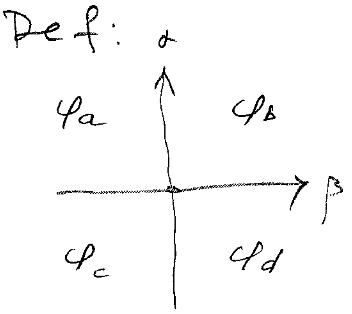
Functional operator: $\forall\, \phi = \phi(A_1 A_2 A_3)$

$$(\mathbb{R}_{123} \circ \phi)(A_1 A_2 A_3) \stackrel{\text{def}}{=} \phi(A_1' A_2' A_3')$$

~~What are~~ Most simple & most important examples:

Electric network transformation

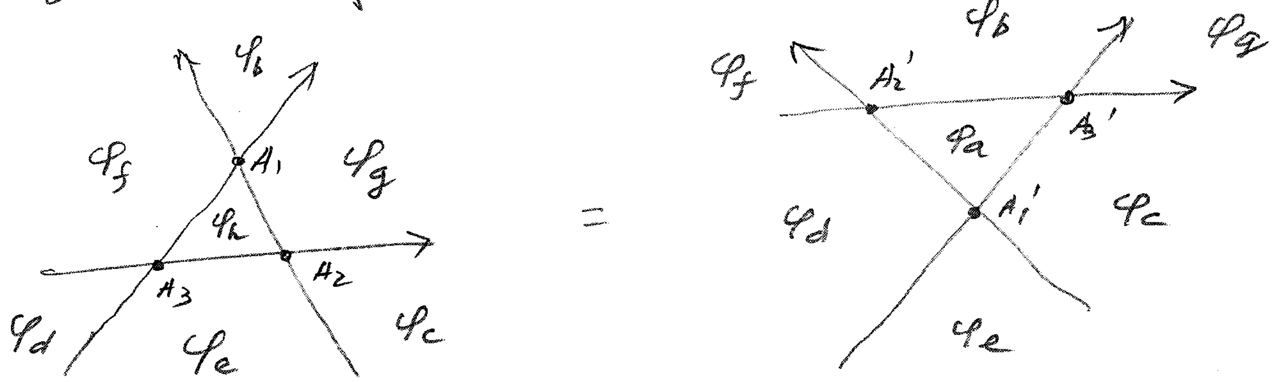
Def:



$$A = W = (u, w, x)$$
$$uw = q^2 wu \; ; \quad x \in \mathbb{C}$$
$$v = x w u, \quad x = w^{-1} v u^{-1} \text{ — center}$$

L.P: $\boxed{|\varphi_a\rangle - q^{-1} u |\varphi_b\rangle - w|\varphi_c\rangle + x w u |\varphi_d\rangle = 0}$

single equation related auxiliary fields surrounding vertex



lhs

$$\begin{cases} \varphi_f - q^{-1} u_1 \varphi_b - w_1 \varphi_h + x_1 w_1 u_1 \varphi_g = 0 & L_1 \\ \varphi_h - q^{-1} u_2 \varphi_g - w_2 \varphi_e + x_2 w_2 u_2 \varphi_c = 0 & L_2 \\ \varphi_f - q^{-1} u_3 \varphi_h - w_3 \varphi_d + x_3 w_3 u_3 \varphi_e = 0 & L_3 \end{cases}$$

rhs

$$\begin{cases} \varphi_b - q^{-1} u_3' \varphi_g - w_3' \varphi_a + x_3 w_3' u_3' \varphi_c = 0 & R_3 \\ \varphi_f - q^{-1} u_2' \varphi_b - w_2' \varphi_d + x_2 w_2' u_2' \varphi_a = 0 & R_2 \\ \varphi_d - q^{-1} u_1' \varphi_a - w_1' \varphi_e + x_1 w_1' u_1' \varphi_c = 0 & R_1 \end{cases}$$

Solve LHS w.r.t. $\varphi_a, \varphi_g, \varphi_e$ } solve RHS w.r.t. $\varphi_h \cdot \varphi_g \, \varphi_e$

$$\begin{pmatrix} \varphi_g \\ \varphi_e \end{pmatrix} = \underbrace{\begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix}}_{\text{LHS}} \begin{pmatrix} \varphi_c \\ \varphi_f \\ \varphi_d \\ \varphi_b \end{pmatrix} = \underbrace{\begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix}}_{\text{RHS}} \begin{pmatrix} \varphi_c \\ \varphi_f \\ \varphi_d \\ \varphi_b \end{pmatrix}$$

LHS ← equate them → RHS

Six of eight equation give immediately the map

$$\boxed{\begin{array}{l}
w_1' = w_2 \Lambda_3\ ; \quad w_2' = \Lambda_3^{-1} w_1 \quad ; \quad w_3' = \Lambda_2^{-1} u_1^{-1} \\
u_1' = \Lambda_2^{-1} w_3\ ; \quad u_2' = \Lambda_1^{-1} u_3 \quad ; \quad u_3' = u_2 \Lambda_1 \\[2ex]
\left\{\begin{array}{l}
\Lambda_1 = u_1^{-1} u_3 - q\, u_1^{-1} w_1 + x_1 w_1 u_2^{-1} \\
\Lambda_2 = \dfrac{x_1}{x_2} u_2^{-1} w_3^{-1} + \dfrac{x_3}{x_2} u_1^{-1} w_2^{-1} - q^{-1} \dfrac{x_1 x_3}{x_2} u_2^{-1} w_2^{-1} \\
\Lambda_3 = w_1 w_3^{-1} - q\, u_3 w_3^{-1} + x_3 w_2^{-1} u_3
\end{array}\right.
\end{array}}$$

! The map is obtained without <u>any</u> exchange relations for primed elements.

! The map is automorphism of $W_1 \otimes W_2 \otimes W_3$

! Two extra relations of eight are then satisfied due to Weyl algebra arithmetics

$\longleftrightarrow$

R- matrix : $\quad W_K' = R_{123} W_K R_{123}^{-1}$

compact representation

$$\hat{u}\hat{w} = q^2 \hat{w}\hat{u}\ , \quad q = e^{i\pi/N} \qquad \left\{\begin{array}{ll} \hat{u} = u X\ , & X|\alpha\rangle = |\alpha\rangle q^{2\alpha} \\ \hat{w} = w Z\ , & Z|\alpha\rangle = |\alpha + 1 \bmod N\rangle \end{array}\right.$$

But $\hat{u}^N = u^N$, $\hat{w}^N = w^N \leftrightarrow$ two more centers

$$\left\{\begin{array}{l}
\hat{w}_1'^N = w_2^N \Lambda_3^N\ ; \quad \hat{w}_2'^N = \Lambda_3^{-N} w_1^N\ ; \quad \hat{w}_3'^N = \Lambda_2^{-N} u_1^{-N} \\
\hat{u}_1'^N = \Lambda_2^{-N} w_3^N\ ; \quad \hat{u}_2'^N = \Lambda_1^{-N} u_3^N\ ; \quad \hat{u}_3'^N = u_2^N \Lambda_1^N \\[2ex]
\Lambda_1^N = u_1^{-N} u_3^N + u_1^{-N} w_1^N + x_1^N w_1^N u_2^{-N} \\
\Lambda_2^N = \dfrac{x_1^N}{x_2^N} u_2^{-N} w_3^{-N} + \dfrac{x_3^N}{x_2^N} u_1^{-N} w_2^{-N} + \dfrac{x_1^N x_3^N}{x_2^N} u_2^{-N} w_2^{-N} \\
\Lambda_3^N = w_1^N w_3^{-N} + u_3^N w_3^{-N} + x_3^N w_2^{-N} u_3^N
\end{array}\right.$$

Hint: $\quad (1-z)(1-q^2 z)\ldots(1-q^{2N-2} z) = 1 - z^N$

Matrix part of the map

$\hat{u}'_k = u'_k X'_k$, $\hat{w}'_k = w'_k Z'_k$ where $u'_k = \sqrt[N]{\hat{u}_k'^{N}}$, $w'_k = \sqrt[N]{\hat{w}_k'^{N}}$

$X'_k R = R X_k$, $Z'_k R = R Z_k$ → recursion relations for matrix elements of R

answer

$$\langle \alpha_1\alpha_2\alpha_3 | R | \beta_1\beta_2\beta_3 \rangle = \delta_{\alpha_2+\alpha_3,\,\beta_2+\beta_3}\, q^{2(\beta_1-\alpha_1)\beta_3}\, \frac{W_{p_1}(\alpha_2-\alpha_1)\, W_{p_2}(\beta_2-\beta_1)}{W_{p_3}(\beta_2-\alpha_1)\, W_{p_4}(\alpha_2-\beta_1)}$$

where $p = (x, y : x^N + y^N = 1)$; $\frac{W_p(n)}{W_p(n-1)} = \frac{y}{1 - x q^{2n}}$; $W_p(0) = 1$

$n \in \mathbb{Z}_N$

$\{p_1, p_2, p_3, p_4\}$:

$$x_1 = \frac{1}{q\varkappa_1}\frac{u_2}{u_1}, \quad x_2 = \frac{\varkappa_2}{q}\frac{u_2'}{u_1'}, \quad x_3 = q^{-2}\frac{u_2'}{u_1}, \quad x_4 = q^{-2}\frac{\varkappa_2}{\varkappa_1}\frac{u_2}{u_1'}$$

$$x_1 x_2 = q^2 x_3 x_4$$

$$\frac{y_3}{y_1} = \varkappa_1 \frac{w_1}{u_3'}, \quad \frac{y_4}{y_1} = \frac{\varkappa_3}{q}\frac{w_3}{w_2}, \quad \frac{y_3}{y_2} = \frac{w_2'}{w_3}$$

$\hat{u}'_k = R \cdot (\mathcal{R}^{(f)} \circ \hat{u}_k) \cdot R^{-1}$ — R is constructed

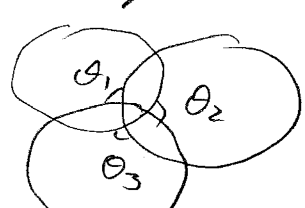
$\mathcal{R}^{(f)} \circ u_k = u'_k$: Functional counterpart

$\left.\begin{array}{l} \hat{u}_k = u_k X_k \\ \hat{w}_k = w_k Z_k \end{array}\right\}$ structure of a bundle

Dynamics of u_k, w_k — complicated essentially 3D model, algebraic curves

Simplest case — no functional counterpart $u'_k = u_k$, $w'_k = w_k$ ↔ set of equations for u_k, w_k in terms of $\varkappa_k$ → spherical triangle

$\varkappa_1^N = \left(\tan\frac{\theta_1}{2}\right)^2$, $\varkappa_2^N = \left(\cot\frac{\theta_2}{2}\right)^2$, $\varkappa_3^N = \left(\tan\frac{\theta_3}{2}\right)^2$

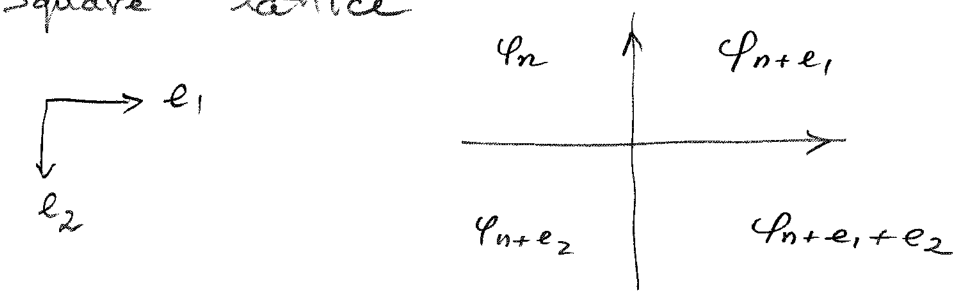


→ R-matrix for ZBB model →

in 2D R-matrix for minimal cyclic reps of $U_q(\widehat{sl}_M)$

Auxiliary problem → auxiliary transfer matrix.

Square lattice

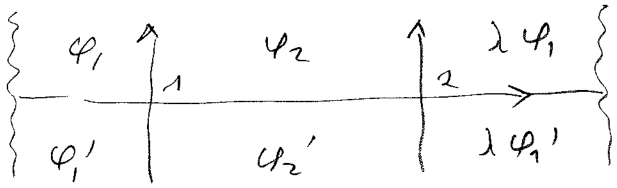


$$J_n \overset{def}{=} \varphi_n - q^{-1} u_n \varphi_{n+e_1} - w_n \varphi_{n+e_2} + \varkappa_n w_n u_n \varphi_{n+e_1+e_2} = 0$$

Boundary conditions

$$\varphi_{n+N_1 e_1} = \lambda \varphi_n \quad ; \quad \varphi_{n+N_2 e_2} = \mu \varphi_n \quad \Big\} \text{ lattice } N_1 \otimes N_2$$

example $N_1 = 2$



$$\begin{cases} \varphi_1 - q^{-1} u_1 \varphi_2 - w_1 \varphi_1' + \varkappa_1 w_1 u_1 \varphi_2' = 0 \\ \varphi_2 - q^{-1} u_2 \lambda \varphi_1 - w_2 \varphi_2' + \varkappa_2 w_2 u_2 \lambda \varphi_1' = 0 \end{cases} \Longleftrightarrow$$

$$(1 - q^{-2}\lambda u_1 u_2)\varphi_1 = (w_1 - q^{-1}\lambda \varkappa_2 w_2 u_1 u_2)\varphi_1' + (q^{-1} w_2 - \varkappa_1 w_1) u_1 \varphi_2'$$

$$(1 - q^{-2}\lambda u_1 u_2)\varphi_2 = \lambda (q^{-1} w_1 - \varkappa_2 w_2) u_2 \varphi_1' + (w_2 - q^{-1}\lambda \varkappa_1 w_1 u_1 u_2)\varphi_2'$$

$$(1 - q^{-2}\lambda u_1 u_2)\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} = w_2 \cdot \begin{pmatrix} w_2^{-1} w_1 - q^{-1}\lambda \varkappa_2 u_1 u_2 & (q^{-1} - \varkappa_1 w_2^{-1} w_1) u_1 \\ \lambda (q^{-1} w_2^{-1} w_1 - \varkappa_2) u_2 & 1 - q^{-1}\lambda \varkappa_1 w_2^{-1} w_1 u_1 u_2 \end{pmatrix}\begin{pmatrix}\varphi_1' \\ \varphi_2'\end{pmatrix}$$

$w_2^{-1} w_1 := w$, $\lambda u_1 u_2 := \lambda$, $u_1 := u$, $\lambda u_2 := \lambda u^{-1}$

$$L^{(\lambda)} = \begin{pmatrix} w - q^{-1}\lambda \varkappa_2 , & (q^{-1} - \varkappa_1 w) u \\ \lambda (q^{-1} w - \varkappa_2) u^{-1}, & 1 - q^{-1}\lambda \varkappa_1 w \end{pmatrix}$$

L-operator for CPM!

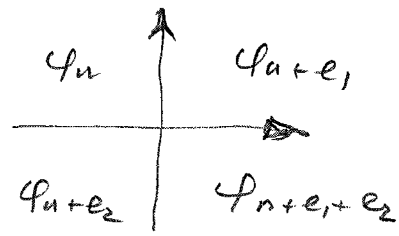
w_2 - multiplier does not commute with $u_1 u_2$!

End of lecture

OK, Aux problem on whole lattice

$$f_n \stackrel{def}{=} \varphi_n - q^{-1} u_n \varphi_{n+e_1} - w_n \varphi_{n+e_2} + \varkappa_n w_n u_n \varphi_{n+e_1+e_2} = 0$$

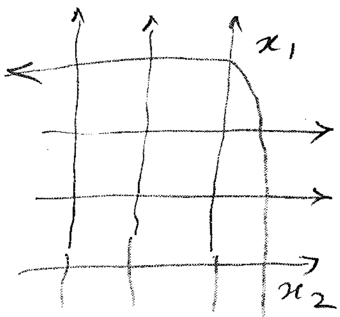
b.c. $\varphi_{n+N_1 e_1} = \lambda \varphi_n$, $\varphi_{n+N_2 e_2} = \mu \varphi_n$



Matrix of coefficients:

$$f_n = \sum_n h_{n,m} \varphi_n \Rightarrow \text{elements from different rows commute}$$

$$J(\lambda,\mu) = \det(h_{n,m}) = \sum_{n,m=0}^{N_2,N_1} \lambda^n \mu^m J_{n,m}$$



$J(\lambda,\mu)\, T(x_1,x_2) = T(x_1,x_2)\, J(\lambda,\mu)$ OK.

$J(\lambda,\mu)$ — "auxiliary transfer matrix"

MYSTERIOUS SURPRISE: $\boxed{J_{n,m} J_{n',m'} = q^{2(nm'-n'm)} J_{n'm'} J_{n,m}}$

$$u_0 = \prod_{n_1} u_{n_1 e_1}, \quad w_0 = \prod_{n_2} w_{n_2 e_2}; \quad J_{n,m} = u_0^n w_0^m q^{-nm} \underbrace{t_{n,m}}_{\text{commute}}$$

<u>Combinatorial statement</u>: QUISM-

$$t_m(\lambda) = \sum_{n=0}^{N_2} \lambda^n t_{n,m}$$

is transfer matrix for m-th fundamental representation of $U_q(\widehat{sl}_{N_1})$ in auxiliary space ($m=0$ & $m=N_1$ are scalars)

$$J(\lambda,\mu) = \sum_{n,m=0}^{N_2,N_1} (\lambda u_0)^n (\mu w_0)^m q^{-nm} t_{n,m}$$

Spectral equation

$$f_n \stackrel{\text{def}}{=} \varphi_n - \bar q^{\,\prime} u_n \varphi_{n+e_1} - w_n \varphi_{n+e_2} + \varkappa_n w_n u_n \varphi_{n+e_1+e_2} = \sum_m h_{n,m}\varphi_m$$

$$J_n \stackrel{\text{def}}{=} \phi_n + u_n^N \phi_{n+e_1} - w_n^N \phi_{n+e_2} + \varkappa_n^N w_n^N u_n^N \phi_{n+e_1+e_2} = \sum_m h^{(N)}_{n,m}\phi_m$$

b.c. $\phi_{n+N_1 e_1} = \lambda^N \phi_n$, $\phi_{n+N_2 e_2} = \mu^N \phi_n$

Theorem

$$\det_{u_0, w_0} J(\lambda,\mu) = \det\left(h^{(N)}_{n,m}\right) = F\left(\lambda^N, \mu^N\right) \qquad (*)$$

- in decomposition wrt $\lambda^N, \mu^N \to$ complete abelian algebra for $t_{n,m}$
- suitable for thermodynamic limit $N_1, N_2 \to \infty$

  for instance, for homogeneous case $u_n^N = -1$, $w_n^N = 1$, $\varkappa_n = \varkappa$

$$F(\lambda^N,\mu^N) = \prod_{n_1,n_2}\left(1 - x e^{2\pi i \frac{n_1}{N_1}} - y e^{2\pi i \frac{n_2}{N_2}} - \varkappa^N x y e^{2\pi i\left(\frac{n_1}{N_1} + \frac{n_2}{N_2}\right)}\right)$$

$$x^{N_1} = \lambda^N, \quad y^{N_2} = \mu^N$$

- In fact, (*) encodes the whole tail of NBAEs

Demonstration for $N_1 = 2$, N_2 - arbitrary

$$J(\lambda,\mu) = \sum_{n,m=0}^{N_2,N_1} (\lambda u_0)^n (\mu w_0)^m t_{n,m} = \underbrace{t_0(\lambda u_0)}_{\alpha} + \mu w_0 t_1(\lambda u_0) + \underbrace{\mu^2 w_0^2 t_2(\lambda u_0)}_{\delta}$$

$$\mu^{-1} w_0^{-1} t_0(\lambda u_0) + t_1(\lambda u_0) + \mu w_0 t_2(\lambda u_0) =$$

$$= \begin{pmatrix} t_1(\lambda) & \mu^{-1} t_0(\lambda) & & & \mu t_2(q^{2(N-1)}\lambda) \\ \mu t_2(\lambda) & t_1(q^2\lambda) & \mu^{-1} t_0(q^2\lambda) & & \\ & \mu t_2(q^2\lambda) & \ddots & \ddots & \\ & & \ddots & \ddots & \\ \mu^{-1} t_0(q^{2(N-1)}\lambda) & & & & t_1(q^{2(N-1)}\lambda) \end{pmatrix}$$