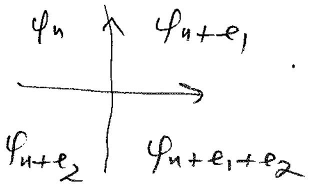
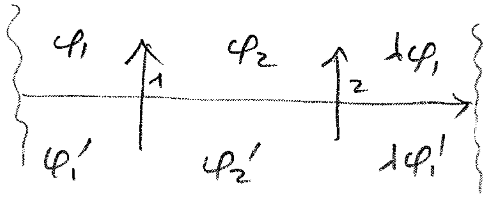
§1. Auxiliary T.M. & spectral equation for Type A



$$j_n \overset{\text{def}}{=} \varphi_n - q^{-1} u_n \varphi_{n+e_1} - w_n \varphi_{n+e_2} + \varkappa_n w_n u_n \varphi_{n+e_1+e_2} = 0 \quad (*)$$

b.c. $\varphi_{n+N_1 e_1} = \lambda \varphi_n$, $\varphi_{n+N_2 e_2} = \mu \varphi_n$



$$(1-q^{-2}\lambda u_1 u_2)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = w_2 \cdot L_{ePM}(w_2^{-1} w_1, u_1, \lambda u_1 u_2)\cdot\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix}$$

solving linear problem step-by-step $\xrightarrow{\text{exclude } \varphi}$ monodromy & transf. mat.

<u>Prescription</u>: solve linear problem

$(*)$: $j_n = \sum_m h_{n,m} \varphi_m = 0$ elements from different rows of $h_{n,m}$ commute! $\Rightarrow$

$$J(\lambda,\mu) = \det(h_{n,m}) = \sum_{n,m=0}^{N_2, N_1} \lambda^n \mu^m J_{nm} \rightarrow \text{aux T.M. commute with layer-to-layer TM}$$

$J_{00} = 1$

Theorem ①

$$J_{nm} J_{n'm'} = q^{2(nm' - n'm)} J_{n'm'} J_{nm} \Rightarrow \exists\, X_0, Z_0 \text{ (mass center)}$$

$$J_{n,m} = q^{-nm} X_0^n Z_0^m t_{nm}, \quad t_{nm} - \text{involutive} \ (\text{\& Hermitian for } \underline{N=2},\ q^{2N}=1)$$

Theorem ②

$$\det_{X_0 Z_0} J(\lambda,\mu) = F(\lambda^N, \mu^N) \leftrightarrows \det(h_{n,m}^{(N)}) : \lambda,\mu \text{ are free!} \quad (**)$$

$$J_n \leftrightarrows \phi_n + u_n^N \phi_{n+e_1} - w_n^N \phi_{n+e_2} + \varkappa_n^N w_n^N u_n^N \phi_{n+e_1+e_2} \leftrightarrows \sum_m h_{n,m}^{(N)} \phi_m$$

b.c. $\phi_{n+N_1 e_1} = \lambda^N \phi_n$, $\phi_{n+N_2 e_2} = \mu^N \phi_n$

$\lambda\mu$ - free $(**)$ - abelian (complete) of $t_{n,m}$

How it works; $N_1 = 2$

$$J(\lambda,\mu) = \sum_{n=0}^{N_2} \underbrace{(\lambda X_0)^n t_{n,0}}_{\alpha(\lambda X_0)} + \mu Z_0 \underbrace{(q\lambda X_0)^n t_{n,1}}_{t_1(\lambda X_0)} + \mu^2 Z_0^2 \underbrace{(q^2 \lambda X_0)^n t_{n,3}}_{\delta(\lambda X_0)}$$

($N_1 > 2$ – all fundamental transfer matrices)

$$\frac{1}{\mu Z_0} J(\lambda,\mu) =$$

$$\begin{pmatrix} t(\lambda) & \mu^{-1}\alpha(\lambda) & & & \mu\delta(q^{-2}\lambda) \\ \mu\delta(\lambda) & t(q^2\lambda) & \mu^{-1}\alpha(q^2\lambda) & & \\ & \mu\delta(q^2\lambda) & t(q^4\lambda) & \ddots & \\ & & \ddots & \ddots & \ddots \\ \mu^{-1}\alpha(q^{-2}\lambda) & & & \ddots & t(q^{2(N-1)}\lambda) \end{pmatrix}$$

μ: $F(\lambda^N, \mu^N) = 0 \Rightarrow \exists Q(\lambda)$ as null vector

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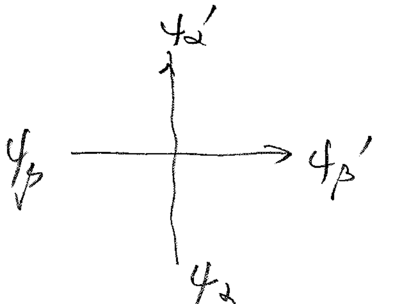
General position

$\det\limits_{X_0 Z_0} J(\lambda,\mu) = F(\lambda^N, \mu^N)$ / universal equation, all tail of NBAE for square lattice in particular

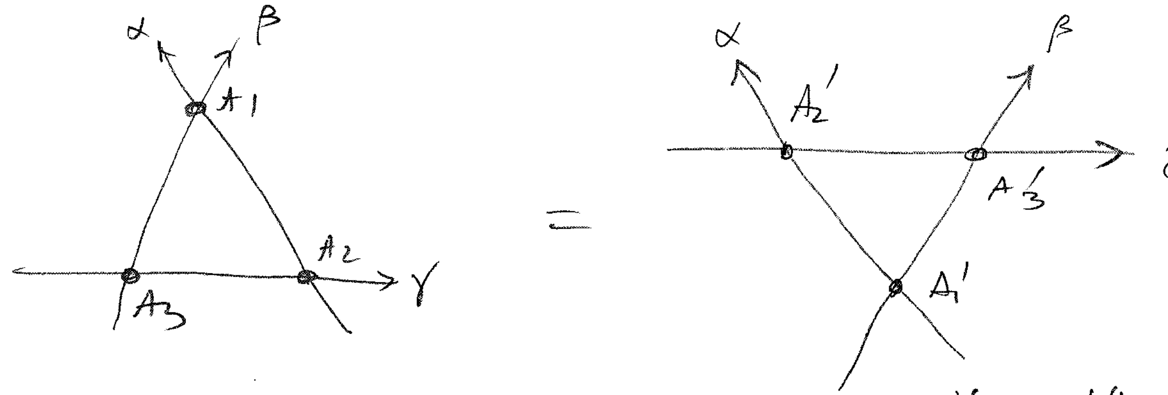
- good for thermodynamic limit ($\lambda\mu$ – arbitrary)
- limitig equations are not integral equations
- Structure of the spectrum is known, there are particle-like excitations, but neither dispersion relation nor S-matrix are not known yet.

§2. Linear problem Type (B)



$$\begin{cases} \psi_\alpha' = \lambda q^N \psi_\alpha + b^+ \psi_\beta \\ \psi_\beta' = -\frac{\lambda\mu}{q} b \psi_\alpha + \mu q^N \psi_\beta \end{cases}$$

$$A = (b, b^+, N; \lambda, \mu) : \ b^+ b = 1 - q^{2N}, \ [N, b] = -b$$



Solution (without any assumption concerning "primed" operators)

$$b_2^{+\prime} = \lambda_1 \mu_3 q^{N_1+N_3} b_2^+ + b_1^+ b_3^+ ; \quad b_2' = \frac{q^2}{\lambda_1\mu_3} q^{N_1+N_3} b_2 + b_1 b_3$$

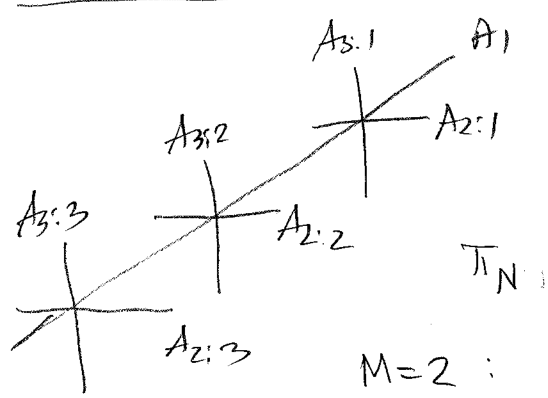
$$b_1^{+\prime} = q^{-N_2'} \frac{\lambda_3}{\lambda_2}\left(q^{N_3} b_1^+ - \frac{\lambda_1\mu_3}{q} q^{N_1} b_2^+ b_3\right), \quad b_1' = q^{-N_2'}\frac{\lambda_2}{\lambda_3}\left(q^{N_3} b_1 - \frac{q}{\lambda_1\mu_3} q^{N_1} b_2 b_3^+\right)$$

$$b_3^{+\prime} = q^{-N_2'}\frac{\mu_1}{\mu_2}\left(q^{N_1} b_3^+ - \frac{\lambda_1\mu_3}{q} q^{N_3} b_1 b_2^+\right), \quad b_3' = q^{-N_2'}\frac{\mu_2}{\mu_1}\left(q^{N_1} b_3 - \frac{q}{\lambda_1\mu_3} q^{N_3} b_1^+ b_2\right)$$

Invariants: $N_1' + N_2' = N_1 + N_2$, $N_2' + N_3' = N_2 + N_3$ — again automorphism

$A' = R A R^{-1}$, Fock space, recursions $\Rightarrow$

$\langle n_1 n_2 n_3 | R | n_1' n_2' n_3' \rangle =$ explicit expression $= R\left(\frac{\lambda_3}{\lambda_2}, \lambda_1\mu_3, \frac{\mu_1}{\mu_2}\right)$



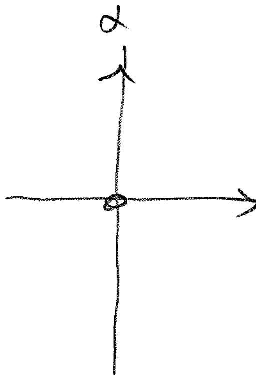
$N_2 = \sum_m N_{2:m}$ $\quad N_3 = \sum_m N_{3:m}$

$R_{A_2, A_3} \in \pi_{N_2 \cdots} \otimes \pi_{N_3 \cdots}$ $(\mathcal{R}\ \widehat{sl}_M)$

π_N : N^{th} symmetric tensors of $\widehat{sl}_M$

$M = 2$: $\pi_N =$ spin $\frac{N}{2}$, $U_q(sl_2)$ – complete!

Linear problem $\leftrightarrow$ free-fermion structure



$$\beta : L_{\alpha\beta}[A] \in \mathrm{End}(V_\alpha \otimes V_\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda q^N & b^+ & 0 \\ 0 & -\frac{\lambda\mu}{q} b & \mu q^N & 0 \\ 0 & 0 & 0 & -\frac{\lambda\mu}{q} \end{pmatrix}$$

$$V_\alpha, V_\beta = \mathbb{C}^2$$

$$L_{\alpha\beta}[A_1]\, L_{\alpha\gamma}[A_2]\, L_{\beta\gamma}[A_3]\, R_{123} = R_{123}\, L_{\beta\gamma}[A_3]\, L_{\alpha\gamma}[A_2]\, L_{\alpha\beta}[A_1]$$

AUX. TM

$$\mathcal{T} = \underset{V_\alpha, V_\beta}{\mathrm{Trace}}\; D_\beta(v) \prod_{n,m}^{\curvearrowright} L_{\alpha;n;\beta;m}[A_{nm}]\, D_\alpha(u) =$$

$$= \sum_{m,n=0}^{M,N} u^m v^n t_{n,m} \quad \text{– involutive}$$

Still

$$N_{*m} = \sum_n N_{nm}$$

$$N_{n*} = \sum_m N_{n,m}$$

conserving charges

$$D_\beta(v) = \bigotimes_m \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}_{V_{\beta;m}} \qquad D_\alpha(u) = \bigotimes_n \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}_{V_{\alpha;n}}$$

$$t_n(u) = \sum_m u^m t_{n,m} \doteq n^{th} \text{ fundamental T.M. of } sl_N$$

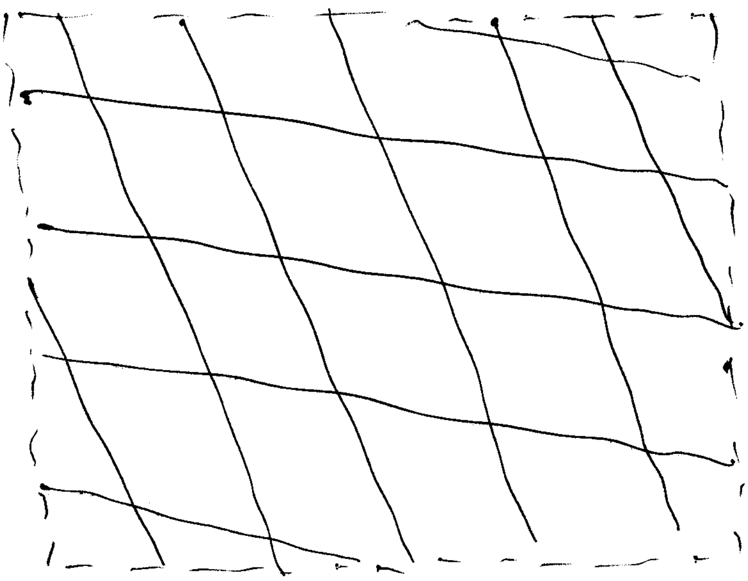
<u>Theorem</u> 3 on square lattice BAE are

$$\sum v^n Q(q^{2n} u)\, t_n((-q)^n u) = 0$$

for some particular values of v $Q(u)$ is polynomial

- TQ equations are not suitable in thermodynamic limit since they have no $M-N$ symmetry
- Too much trivial charges

3D lattice Bose gas



Bi-spiral lattice
Winding numbers N+M
Framework of $L_{\alpha\beta}[A]$
Coordinate BA

$\omega_j = e^{i p_{x,j}}$ $\quad \Omega_j = e^{i p_{y,j}}$ $\quad (p_x, p_y) = \vec{p}$ – momentum

$\mathcal{H} = \sum_j \left(2 - \cos p_{x,j} - \cos p_{y,j}\right)$ $\quad$ K particles

$$G_{jk}(\omega) = \frac{q^{-1}\omega_k - q\,\omega_j}{\omega_k - \omega_j}$$

$(1, 2, \dots, K) = I_a \cup I_{K-a}$ (I_a is length $= a$ subsequence)

$S_a(\omega) = \sum_{I_a} \prod_{j \in I_a} \omega_j$ (symmetrical polynomials)

$$P_a\left(\omega, \omega^M\right) = \sum_{I_a} \left(\prod_{j \in I_a} \omega_j^M \prod_{l \in I_{K-a}} G_{lj}(\omega) \right)$$

BAE:
$$\begin{cases} S_a(\Omega) = P_a(\omega, \omega^M) \\ S_a(\omega) = P_a(\Omega, \Omega^N) \end{cases} \quad \forall a = 1 \dots K$$

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