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On the Construction of Determinant representations of Correlation Functions for the Supersymmetric t - J Models

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Abstract

In this talk, we will review the recent progress on the construction of determinant representations of correlation functions for the supersymmetric t - J model. The main approach of the work is based on the Drinfeld twists (F -basis). Working in the F -basis, we find the entries of the monodromy matrix and the Bethe states of the model can be expressed by using symmetric formulaes.

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Part I: Background and main method

I.1 Background

For the algebra $gl(2)$ case

- **1981:** Gaudin *et al* proposed a hypothesis that the norm of the coordinate eigenstates for the XXZ spin chain model can be given by some Jacobians.
- **1982:** Korepin proved the hypothesis. Then Korepin and his collaborators found that the correlation functions of the model can be represented by using determinants.
- **1992:** Jimbo *et al* obtained the correlation functions for one dimensional integrable models by using the technique of q-deformed vertex operators.
- **1996:** Maillet and his group proposed the Drinfeld twist approach to construct the determinant representations for correlation functions of the XXZ model.

Progress for the higher rank (super)algebra case

- **1986** Reshetikhin found a recursive formula for the scalar product of the integrable model related to $gl(3)$.
- **2001:** Flume and his group constructed the Drinfeld twist for the integrable models related to the algebra $gl(n)$. With the help of the approach, they found symmetric expressions of the Bethe states for the model.
- **2004:** Our group generalized the Drinfeld twist approach to the supersymmetric fermion models associated with the superalgebraic $U_q(gl(m|n))$. By means of the approach, we gave symmetric expressions for Bethe states of the integrable fermion model. The results led directly to the determinant representations of the correlation functions for the supersymmetric t - J model.

I.2 Main ideas

In general, a k -point correlation function is defined by

$$F_n^{\epsilon^1, \dots, \epsilon^k} = \langle \Omega_N(\{\mu_j\}) | \epsilon_{i_1}^1 \dots \epsilon_{i_k}^k | \Omega_N(\{\lambda_j\}) \rangle,$$

where $\epsilon_{i_j}^j$'s stand for the local fermion generators of the superalgebras, and the lower indices i_j indicate the positions of the fermion operators.

According to the relations proposed by Kitanine *et al* and Göhmann *et al*, the local fermion operators can be represented by using entries of monodromy matrices. For example: For the supersymmetric t - J model, the fermion operator $(1 - n_{\kappa,\uparrow})c_{\kappa,\downarrow}^\dagger$ can be given by

$$(1 - n_{\kappa,\uparrow})c_{\kappa,\downarrow}^\dagger = \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot C_1(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad ,$$

where $t(\xi)$ is the transfer matrix. Therefore the correlation function becomes

$$F_n^{\epsilon^1, \dots, \epsilon^k} = f(\xi, \lambda, \mu) \langle \Omega_N(\{\mu_j\}) | X(\xi'_1) \dots X(\xi'_k) | \Omega_N(\{\lambda_j\}) \rangle ,$$

where $f(\xi, \lambda, \mu)$ is from the eigenvalues of the transfer matrices, $\{\xi\} = \{\xi'\}$ and $X(\xi)$ stands for the entries of the monodromy matrices.

Now as an example, we investigate the supersymmetric t - J model.

Considering the monodromy matrix of the model,

$$\begin{aligned} T(\lambda) &= R_{0N}(\lambda - \xi_N) \dots R_{01}(\lambda - \xi_1) \\ &= \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix}_{(0)}, \end{aligned}$$

the correlation functions can be given by

$$\begin{aligned} &F_n^{\epsilon^1, \dots, \epsilon^k} \\ &= \sum_{d'_1, \dots, d'_n; d_1, \dots, d_n=1,2} (\langle \Omega_N^{(1)}(\{\mu_j^{(1)}\}) \rangle^{d'_1, \dots, d'_n} (|\Omega_N^{(1)}(\{\mu_j^{(1)}\})\rangle)^{d_1, \dots, d_n} \\ &\quad \times f(\xi, \lambda, \mu) \langle 0 | B_{d'_n}(\mu_n) \dots B_{d'_1}(\mu_1) X(\xi'_1) \dots X(\xi'_k) \\ &\quad \times C_{d_1}(\lambda_1) \dots C_{d_n}(\lambda_n) | 0 \rangle. \end{aligned} \tag{1}$$

Difficulties: The creation (annihilation) operators C_i (B_i) and other entries of the monodromy matrix are very complex. Therefore it is very difficult to compute the correlation functions for the model.

The idea to solve the problem: One of the possible ideas to solve the problem is to look for a proper basis with which we can simplify the creation operators. For the main part of (1), we have

$$\langle 0 | B_{d'_n}(\mu_n) \dots B_{d'_1}(\mu_1) X(\xi_1) \dots X(\xi_k) C_{d_1}(\lambda_1) \dots C_{d_n}(\lambda_n) | 0 \rangle$$

$$\begin{aligned}
&= \langle 0|B_{d'_n}(\mu_n) \dots B_{d'_1}(\mu_1) \cdot F^{-1}F \cdot \\
&\quad \times X(\xi_1) \dots X(\xi_k)C_{d_1}(\lambda_1) \dots C_{d_n}(\lambda_n)|0\rangle \\
&= g(\{\xi\})\langle \tilde{0}|\tilde{B}_{d'_n}(\mu_n) \dots \tilde{B}_{d'_1}(\lambda_1) \\
&\quad \times \tilde{X}(\xi_1) \dots \tilde{X}(\xi_k)\tilde{C}_{d_1}(\lambda_1) \dots \tilde{C}_{d_n}(\lambda_n)|\tilde{0}\rangle.
\end{aligned}$$

Here $\tilde{X} = FXF^{-1}$, $|\tilde{0}\rangle = F|0\rangle$. Thus if we can find a suitable F -basis, then we can make the computation easier than before. In the next, we will construct such a F -basis.

Part II: Drinfeld twists

II.1 N-site R -matrix

We firstly define a useful notation: N -site R -matrix $R_{1\dots N}^\sigma$. Let σ be an element of the permutation group \mathcal{S}_N . The N -site R -matrix is defined as follows.

Definition 1 *One can define a mapping from \mathcal{S}_N to $\text{End}(V^{\otimes N})$ which associate in a unique way an element $R_{1\dots N}^\sigma \in \text{End}(V^{\otimes N})$ to any element σ of the permutation group \mathcal{S}_N . The mapping has the following composition law*

$$R_{1\dots N}^{\sigma\sigma'} = \mathcal{P}^\sigma R_{1\dots N}^{\sigma'} (\mathcal{P}^\sigma)^{-1} R_{1\dots N}^\sigma = R_{\sigma(1)\dots\sigma(N)}^{\sigma'} R_{1\dots N}^\sigma, \quad \forall \sigma, \sigma' \in \mathcal{S}_N,$$

where $R_{1\dots N}^{\sigma=(i,i+1)} = R_{i(i+1)}$ and \mathcal{P}^σ is the \mathbb{Z}_2 -graded permutation operator

$$\mathcal{P}^\sigma |i_1\rangle_{(1)} \cdots |i_N\rangle_{(N)} = |i_1\rangle_{(\sigma(1))} \cdots |i_N\rangle_{(\sigma(N))}.$$

By using the composition law defined in the definition, we can decompose the N -site R -matrix to elementary R -matrix of the $gl(m|n)$

model

$$\begin{aligned}
R_{12}(\lambda_1, \lambda_2) &= R_{12}(\lambda_1 - \lambda_2) \\
&= c_{12} \sum_{i=1}^m e_{i,i} \otimes e_{i,i} + \sum_{i=m+1}^{m+n} e_{i,i} \otimes e_{i,i} + a_{12} \sum_{i \neq j=1}^{m+n} e_{i,i} \otimes e_{j,j} \\
&\quad + b_{12} \sum_{i>j=1}^{m+n} (-1)^{[j]} e_{i,j} \otimes e_{j,i} + b_{12} \sum_{j>i=1}^{m+n} (-1)^{[j]} e_{i,j} \otimes e_{j,i},
\end{aligned}$$

where

$$\begin{aligned}
a_{12} = a(\lambda_1, \lambda_2) &\equiv \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \eta}, \\
b_{12} = b(\lambda_1, \lambda_2) &\equiv \frac{\eta}{\lambda_1 - \lambda_2 + \eta}, \\
c_{12} = c(\lambda_1, \lambda_2) &\equiv \frac{\lambda_1 - \lambda_2 - \eta}{\lambda_1 - \lambda_2 + \eta}.
\end{aligned}$$

II.2 Factorizing F-matrices and their inverses

Definition 2 *The N -site F-matrix $F_{1\dots N}(\xi_1, \dots, \xi_N)$ is an operator in $\text{End}(V^{\otimes N})$ and satisfies the following three properties:*

I.) lower-triangularity; II.) non-degeneracy; III.) factorization, namely,

$$F_{\sigma(1)\dots\sigma(N)}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) R_{1\dots N}^\sigma = F_{1\dots N}(\xi_1, \dots, \xi_N), \quad \forall \sigma \in \mathcal{S}_N.$$

Proposition 1 *The N -site factorizing F -matrix for the $gl(m|n)$ supersymmetric model, given by*

$$\begin{aligned} F_{1\dots N} &\equiv F_{1\dots N}(\xi_1, \dots, \xi_N) \\ &= \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(N)}=1}^{m+n} \sum^* \prod_{j=1}^N P_{\sigma(j)}^{\alpha_{\sigma(j)}} S(\sigma, \alpha_\sigma) R_{1\dots N}^\sigma, \end{aligned}$$

satisfies the properties I, II and III in the definition 2.

Here, P_i^α is the embedding of the project operator P^α in the i -th space with $(P^\alpha)_{kl} = \delta_{kl} \delta_{k\alpha}$, $S(\sigma, \alpha_\sigma)$ is a c-number function of σ, α_σ and the element c_{ij} of the R-matrix

$$S(\sigma, \alpha_\sigma) \equiv \exp \left\{ \frac{1}{2} \sum_{l>k=1}^N \left(1 - (-1)^{[\alpha_{\sigma(k)}]} \right) \delta_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} \ln(1 + c_{\sigma(k)\sigma(l)}) \right\}$$

and the sum \sum^* is defined by

$$\alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)} \quad (\sigma(i+1) > \sigma(i));$$

$$\alpha_{\sigma(i+1)} > \alpha_{\sigma(i)} \quad (\sigma(i+1) < \sigma(i)).$$

For example, when $N = 2$, for supersymmetric t - J model, the R -matrix and F -matrix are as follows.

$$R_{12} = \begin{pmatrix} c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & -b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 & 0 & 0 & b_{12} & 0 & 0 \\ 0 & -b_{12} & 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{12} & 0 & b_{12} & 0 \\ 0 & 0 & b_{12} & 0 & 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{12} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F_{12} = \begin{pmatrix} 1 + c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_{12} & 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + c_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{12} & 0 & 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{12} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The non-degenerate property of the F -matrix implies that we can find the inverse matrix $F_{1\dots N}^{-1}$.

Proposition 2 *The inverse of the F -matrix is given by*

$$F_{1\dots N}^{-1} = F_{1\dots N}^* \prod_{i < j} \Delta_{ij}^{-1},$$

where

$$F_{1\dots N}^* = \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(N)} = 1}^{m+n} S(\sigma, \alpha_\sigma) R_{\sigma(1\dots N)}^{\sigma^{-1}} \prod_{j=1}^N P_{\sigma(j)}^{\alpha_{\sigma(j)}},$$

and

$$[\Delta_{ij}]_{\alpha_i \alpha_j}^{\beta_i \beta_j} = \delta_{\alpha_i \beta_i} \delta_{\alpha_j \beta_j} \begin{cases} a_{ij} & \text{if } \alpha_i > \alpha_j \\ a_{ji} & \text{if } \alpha_i < \alpha_j, \\ 1 & \text{if } \alpha_i = \alpha_j = m+1, \dots, m+n, \\ 4a_{ij} & \text{if } \alpha_i = \alpha_j = 1, \dots, m. \end{cases}$$

Here the sum \sum^{**} is taken over all possible α_i which satisfies the following non-increasing constraints:

$$\alpha_{\sigma(i+1)} \leq \alpha_{\sigma(i)} \quad (\sigma(i+1) < \sigma(i)); \quad \alpha_{\sigma(i+1)} < \alpha_{\sigma(i)} \quad (\sigma(i+1) > \sigma(i)).$$

II.3 Entries of monodromy matrix in the F -basis

The non-degeneracy of the F -matrix means that its column vectors also form a complete basis of the quantum space $V^{\otimes N}$, which is called

the F -basis. In this section, we study the generators of $gl(m|n)$ and the elements of the monodromy matrix in the F -basis.

Introduce the generators in the F -basis: $\tilde{E}_{i,j} = F_{1\dots N} E_{i,j} F_{1\dots N}^{-1}$, $(i, j = 1, \dots, m+n)$, where $E_{i,j} = E_{(1)}^{i,j} + \dots + E_{(N)}^{i,j}$. Then,

Theorem 1 *In the F -basis the Cartan and the simple generators of $gl(m|n)$ are given by*

$$\begin{aligned}\tilde{E}_{i,i} &= E_{i,i} = \sum_{k=1}^N E_{(k)}^{i,i}, \quad i = 1, \dots, m+n, \\ \tilde{E}_{j,j+1} &= \sum_{k=1}^N E_{(k)}^{j,j+1} \otimes_{\gamma \neq k} G_{(\gamma)}^{j,j+1}(k, \gamma), \quad j = 1, \dots, m+n-1, \\ \tilde{E}_{j+1,j} &= \sum_{k=1}^N E_{(k)}^{j+1,j} \otimes_{\gamma \neq k} G_{(\gamma)}^{j+1,j}(k, \gamma), \quad j = 1, \dots, m+n-1.\end{aligned}$$

Here the diagonal matrices $G_{(j)}^{\gamma, \gamma \pm 1}(i, j)$ are:

- For $1 < \gamma + 1 \leq m$,

$$\begin{aligned}(G_{(j)}^{\gamma, \gamma+1}(i, j))_{kl} &= \delta_{kl} \begin{cases} 2, & k = \gamma, \\ (2a_{ij})^{-1}, & k = \gamma + 1, \\ 1, & \text{otherwise,} \end{cases} \\ (G_{(j)}^{\gamma+1, \gamma}(i, j))_{kl} &= \delta_{kl} \begin{cases} 2, & k = \gamma + 1, \\ (2a_{ji})^{-1}, & k = \gamma, \\ 1, & \text{otherwise,} \end{cases}\end{aligned}$$

- For $\gamma = m$,

$$(G_{(j)}^{\gamma, \gamma+1}(i, j))_{kl} = \delta_{kl} \begin{cases} 2, & k = \gamma, \\ 1, & k = \gamma + 1, \\ 1, & \text{otherwise,} \end{cases}$$

$$(G_{(j)}^{\gamma+1, \gamma}(i, j))_{kl} = \delta_{kl} \begin{cases} (2a_{ji})^{-1}, & k = \gamma, \\ (a_{ji})^{-1}, & k = \gamma + 1, \\ 1, & \text{otherwise,} \end{cases}$$

- For $1 + m \leq \gamma < m + n$,

$$(G_{(j)}^{\gamma, \gamma+1}(i, j))_{kl} = \delta_{kl} \begin{cases} (a_{ij})^{-1}, & k = \gamma, \\ 1, & \text{otherwise,} \end{cases}$$

$$(G_{(j)}^{\gamma+1, \gamma}(i, j))_{kl} = \delta_{kl} \begin{cases} (a_{ji})^{-1}, & k = \gamma + 1, \\ 1, & \text{otherwise.} \end{cases}$$

Now let us study the entries of monodromy matrices in the F -basis. We first see the elements $D(\lambda) \equiv T_{m+n, m+n}(\lambda)$ for the $gl(m|n)$ model.

Proposition 3 $\tilde{D}(\lambda) = F_{1\dots N} D(\lambda) F_{1\dots N}^{-1}$ is a diagonal matrix given by

$$\tilde{D}(\lambda) = \otimes_{i=1}^N \text{diag}(a_{0i}, \dots, a_{0i}, 1)_{(i)}.$$

Then with the help of Theorem 1 and Proposition 3, we may compute all other creation (annihilation) operators of the fermion

systems:

$$\begin{aligned}\tilde{C}_{m+n-l}(\lambda) &\equiv FC_{m+n-l}(\lambda)F^{-1} = [\tilde{E}_{m+n-l,m+n}, \tilde{D}(\lambda)], \\ \tilde{B}_{m+n-l}(\lambda) &\equiv FB_{m+n-l}(\lambda)F^{-1} \\ &= (-1)^{[m+n]+[m+n-l]}[\tilde{D}(\lambda), \tilde{E}_{m+n,m+n-l}],\end{aligned}$$

where C_{m+n-l} and B_{m+n-l} are the entries of the monodromy matrix $T_{m+n,m+n-l}$ and $T_{m+n-l,m+n}$, respectively.

We then have for some special values of m and n .

- For $m = 2, n = 0$, i.e. the $gl(2|0)$ case:

$$\begin{aligned}\tilde{C}_1(\lambda) &= - \sum_{i=1}^N b_{0i} E_{(i)}^{1,2} \otimes_{j \neq i} \text{diag} (2a_{0j}, c_{0j}(2a_{ij})^{-1})_{(j)}. \\ \tilde{B}_1(\lambda) &= - \sum_{i=1}^N b_{0i} E_{(i)}^{2,1} \otimes_{j \neq i} \text{diag} (a_{0j}(2a_{ji})^{-1}, 2c_{0j})_{(j)};\end{aligned}$$

- For $m = 2, n = 1$, i.e. the $gl(2|1)$ case,

$$\begin{aligned}\tilde{C}_2(\lambda) &= \sum_{i=1}^N b_{0i} E_{(i)}^{2,3} \otimes_{j \neq i} \text{diag} (a_{0j}, 2a_{0j}, 1)_{(j)}, \\ \tilde{B}_2(\lambda) &= - \sum_{i=1}^N b_{0i} E_{(i)}^{3,2} \otimes_{j \neq i} \text{diag} (a_{0j}, a_{0j}(2a_{ji})^{-1}, a_{ji}^{-1})_{(j)},\end{aligned}$$

$$\begin{aligned}
\tilde{C}_1(\lambda) &= \sum_{i=1}^N b_{0i} E_{(i)}^{1,3} \otimes_{j \neq i} \text{diag} (2a_{0j}, a_{0j}(a_{ij})^{-1}, 1)_{(j)} \\
&+ \sum_{i \neq j=1}^N \frac{a_{0i} b_{0j} b_{ij}}{a_{ij}} E_{(i)}^{1,2} E_{(j)}^{2,3} \otimes_{k \neq i,j} (2a_{0k}, a_{0k}(a_{ik})^{-1}, 1)_{(k)}. \\
\tilde{B}_1(\lambda) &= - \sum_{i=1}^N b_{0i} E_{(i)}^{3,1} \otimes_{j \neq i} a_{ji}^{-1} \text{diag} (a_{0j}/2, a_{0j}, 1)_{(j)} \\
&- \sum_{i \neq j=1}^N \frac{a_{0j} b_{0i} b_{ji}}{a_{ji}} E_{(i)}^{3,2} E_{(j)}^{2,1} \\
&\otimes_{k \neq i,j} (a_{0k}(2a_{kj})^{-1}, a_{0k}(a_{ki})^{-1}, a_{ki})_{(k)}.
\end{aligned}$$

Here $x_{0k} \equiv x(\lambda, \xi)$ with x_{ij} stands for a_{ij}, b_{ij} .

Remark: One sees that working in the F -basis, the creation and annihilation operators take completely symmetric forms. Without this basis, the original creation (annihilation) operators are very difficult to give such simple forms.

Part III: Determinant representations of correlation functions

Note: In the following, as an example, we study the correlation functions for the supersymmetric t - J model.

To compute the correlation functions, there are three steps: Bethe states in the F -basis \longrightarrow Scalar products \longrightarrow Correlation functions.

III.1 The Bethe states in the F -basis

- **Definitions of Bethe states**

In the framework of the algebraic ansatz, the Bethe states (eigenstates) are constructed by acting the creation operators $C_i(\lambda)$ (or the annihilation operators $B_i(\lambda)$) on the pseudo-vacuum state.

Definition 3 *Let $|0\rangle$ be the pseudo-vacuum state of the N -fold quantum tensor space $V^{\otimes N}$, and $|0\rangle^{(1)}$ be the pseudo-nested-vacuum*

state of the n_1 -fold nested quantum tensor space $(V^{(1)})^{\otimes n_1}$, i.e.,

$$|0\rangle = \bigotimes_{i=1}^N \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{(i)}, \quad |0\rangle^{(1)} = \bigotimes_{j=1}^{n_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(j)}.$$

The Bethe state of the q -deformed supersymmetric t - J model is then defined by

$$|\Omega_N(\{\lambda_j\})\rangle = \sum_{d_1 \dots d_{n_1}} (\Omega_{n_1}^{(1)})^{d_1 \dots d_{n_1}} C_{d_1}(\lambda_1) \dots C_{d_{n_1}}(\lambda_{n_1}) |0\rangle, \quad (1)$$

where $\lambda_1 \neq \dots \neq \lambda_{n_1}$, $d_i = 1, 2$, $(\Omega_{n_1}^{(1)})^{d_1 \dots d_{n_1}}$ is a component of the nested Bethe state $|\Omega\rangle^{(1)}$ via

$$|\Omega_{n_1}(\{\lambda_j^{(1)}\})\rangle^{(1)} = C_1^{(1)}(\lambda_1^{(1)}) \dots C_1^{(1)}(\lambda_{n_2}^{(1)}) |0\rangle^{(1)} \quad (\lambda_1^{(1)} \neq \dots \neq \lambda_{n_2}^{(1)}), \quad (2)$$

and $C^{(1)}$, the creation operator of the nested $gl(2)$ system, is the lower-triangular entry of the nested monodromy matrix $T^{(1)}$

$$\begin{aligned} T^{(1)}(\lambda^{(1)}) &= r_{0n_1}(\lambda^{(1)} - \lambda_{n_1}) r_{0n_1-1}(\lambda^{(1)} - \lambda_{n_1-1}) \dots r_{01}(\lambda^{(1)} - \lambda_1) \\ &\equiv \begin{pmatrix} A^{(1)}(\lambda^{(1)}) & B_1^{(1)}(\lambda^{(1)}) \\ C_1^{(1)}(\lambda^{(1)}) & D^{(1)}(\lambda^{(1)}) \end{pmatrix}_{(0)} \end{aligned}$$

with $r_{12}(\lambda_1, \lambda_2) \equiv r_{12}(\lambda_1 - \lambda_2) = c_{12}(e_{1,1} \otimes e_{1,1} + e_{2,2} \otimes e_{2,2}) + a_{12}(e_{1,1} \otimes e_{2,2} + e_{2,2} \otimes e_{1,1}) - b_{12}e_{2,1}(\otimes e_{1,2} + e_{1,2} \otimes e_{2,1})$.

Similarly, we can also define the dual Bethe state $\langle \Omega_N |$.

Definition 4 *With the help of the dual pseudo-vacuum state $\langle 0|$ and the dual pseudo-nested-vacuum state $\langle 0|^{(1)}$, the dual Bethe state is defined by*

$$\langle \Omega_N(\{\mu_j\})| = \sum_{f_{n_1}, \dots, f_1} (\Omega^{(1)})^{f_{n_1} \dots f_1} \langle 0| B_{f_{n_1}}(\mu_{n_1}) \dots B_{f_1}(\mu_1)$$

where $\mu_{n_1} \neq \dots \neq \mu_1$, $(\Omega^{(1)})^{f_{n_1} \dots f_1}$ is a component of the dual nested Bethe state $\langle \Omega|^{(1)}$

$$\langle \Omega_{n_1}(\{\mu_j^{(1)}\})|^{(1)} = \langle 0|^{(1)} B_1^{(1)}(\mu_{n_2}^{(1)}) \dots B_1^{(1)}(\mu_1^{(1)}) (\mu_{n_2}^{(1)} \neq \dots \neq \mu_1^{(1)}).$$

- **Useful properties of the Bethe state**

- A. The exchange symmetry:**

Let $\sigma = \sigma_{i_1} \dots \sigma_{i_k}$ be any element of the permutation group \mathcal{S}_{n_1} with σ_{i_j} be elementary permutations $\sigma_{i_j}(i_j, i_j + 1) = (i_j + 1, i_j)$. We define the following exchange operator $\hat{f}_\sigma = \hat{f}_{\sigma_{i_1}} \dots \hat{f}_{\sigma_{i_k}}$ for the Bethe state $|\Omega_N(\{\lambda_j\})\rangle$ of the supersymmetric t - J model, $\hat{f}_\sigma |\Omega_N(\{\lambda_j\})\rangle = |\Omega_N(\{\lambda_{\sigma(j)}\})\rangle$. Then one may prove, under the action of the exchange operator, we have the following proposition:

Proposition 4 *The Bethe state $|\Omega_N(\{\lambda_j\})\rangle$ of the q -deformed supersymmetric t - J model satisfies the following exchange symmetry*

$$\hat{f}_\sigma |\Omega_N(\{\lambda_j\})\rangle = \frac{1}{c_{1\dots\alpha}^\sigma} |\Omega_N(\{\lambda_j\})\rangle,$$

where $c_{1\dots n_1}^\sigma$ has the decomposition law $c_{1\dots n_1}^{\sigma'\sigma} = c_{\sigma'(1\dots n_1)}^\sigma c_{1\dots n_1}^{\sigma'}$, and $c_{1\dots n_1}^{\sigma_i} = c_{i\ i+1}^\sigma \equiv c(\lambda_i, \lambda_{i+1})$ for an elementary permutation σ_i .

B.Faddev-Zamolodchikov relation

For the creation operators $C_i(v)$ and $C_j(u)$, we have the commutation relation

$$C_i(v)C_j(u) = -\frac{1}{a(u, v)}C_j(u)C_i(v) + \frac{b(u, v)}{a(u, v)}C_j(v)C_i(u).$$

Then together with the exchange symmetry, we may rewrite the Bethe states of supersymmetric t - J model as

$$\begin{aligned} & \Omega_N^{p1}(v_1, \dots, v_n) \\ &= \sum_{d_1, \dots, d_n} (\Omega^{(1)})^{d_1, \dots, d_n} C_{d_1}(\lambda_1) \dots C_{d_n}(\lambda_n) |0\rangle \\ &= \frac{1}{p!(n-p)!} \sum_{\sigma \in \mathcal{S}_n} c_{1\dots n}^\sigma (\Omega_n^{(1), \sigma})^{11\dots 12\dots 2} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{k=1}^p \prod_{l=p_1+1}^n (-a^{-1}(v_{\sigma(l)}, v_{\sigma(k)})) \\
& \times C_2(v_{\sigma(p+1)}) \dots C_2(v_{\sigma(n)}) C_1(v_{\sigma(1)}) \dots C_1(v_{\sigma(p)}) |0\rangle,
\end{aligned}$$

where $d_i = 1, 2$ ($i = 1, \dots, n$) and $(\Omega_n^{(1),\sigma})^{11\dots 12\dots 2} \equiv (\hat{f}_\sigma \Omega_n^{(1)})^{11\dots 12\dots 2}$.

Here, without loss of generality, we will only concentrate on the Bethe state with the **quantum number** p which indicates the number of $d_i = 2$.

C. Eigenstates:

The diagonalization of the transfer matrix $t(\lambda)$ leads to the following theorem:

Theorem 2 *The Bethe states $|\Omega_N(\{\lambda_j\})\rangle$ are eigenstates of the transfer matrix $t(\lambda)$ if the spectral parameters λ_j ($j = 1, \dots, n_1$) satisfy the Bethe ansatz equations (BAE)*

$$\prod_{k=1}^N a(\lambda_j, \xi_k) \prod_{l=1}^{n_2} a^{-1}(\lambda_j, \lambda_l^{(1)}) = 1 \quad (j = 1, \dots, n)$$

and the nested Bethe ansatz equations (NBAE)

$$\prod_{j=1}^{n_1} a(\lambda_j, \lambda_l^{(1)}) \prod_{k=1, \neq l}^{n_2} \frac{a(\lambda_l^{(1)}, \lambda_k^{(1)})}{a(\lambda_k^{(1)}, \lambda_l^{(1)})} = 1 \quad (l = 1, \dots, n_2).$$

The eigenvalues $\Lambda(\lambda, \{\lambda_k\}, \{\lambda_j^{(1)}\})$ of the transfer matrix $t(\lambda)$ are given by

$$\begin{aligned} & \Lambda(\lambda, \{\lambda_k\}, \{\lambda_j^{(1)}\}) \\ &= \prod_{i=1}^N a(\lambda, \xi_i) \prod_{j=1}^{n_1} \frac{1}{a(\lambda, \lambda_j)} \Lambda^{(1)}(\lambda) + \prod_{j=1}^{n_1} \frac{1}{a(\lambda_j, \lambda)}, \end{aligned}$$

where $\Lambda^{(1)}(\lambda)$ is the eigenvalues of the nested transfer matrix $t^{(1)}(\lambda) = \text{str}_0 T^{(1)}(\lambda)$

$$\Lambda^{(1)}(\lambda) = - \prod_{j=1}^{n_1} \frac{a(\lambda, \lambda_j)}{a(\lambda_j, \lambda)} \prod_{k=1}^{n_2} \frac{1}{a(\lambda, \lambda_k^{(1)})} - \prod_{j=1}^{n_1} a(\lambda, \lambda_j) \prod_{k=1}^{n_2} \frac{1}{a(\lambda^{(1)}, \lambda)}.$$

One easily checks that this theorem also holds for the dual Bethe state $\langle \Omega_N(\{\mu_j\}) |$ defined by (3) if we change the spectral parameters λ_j and $\lambda_j^{(1)}$ in (3)-(3) to μ_j and $\mu_j^{(1)}$, respectively.

• The Bethe state in the F -basis

Acting the associated F -matrix on the pseudo-vacuum state $|0\rangle$, one finds that the pseudo-vacuum state is invariant. It is due to the fact that only the term with all roots equal to 3 will produce non-zero results. Therefore, the Bethe state (1) in the F -basis $|\tilde{\Omega}_N(\{\lambda_j\})\rangle \equiv$

$F_{1\dots N}|\Omega_N(\{\lambda_j\})\rangle$ can be written as

$$|\tilde{\Omega}_N(\{\lambda_j\})\rangle = \sum_{d_1\dots d_{n_1}} (\Omega_{n_1}^{(1)})^{d_1\dots d_{n_1}} \tilde{C}_{d_1}(\lambda_1) \dots \tilde{C}_{d_{n_1}}(\lambda_{n_1})|0\rangle.$$

Here we will use the notation $|\tilde{\Omega}_N(\{\lambda_j\}_{(p,n)})\rangle$ with the subscript pair (p, n) to denote a Bethe state which has quantum number p and has n spectral parameters.

Proposition 5 *The Bethe state of the supersymmetric t - J model can be represented in the F -basis by*

$$\begin{aligned} |\tilde{\Omega}_N(\{\lambda_j\}_{(p,n)})\rangle &= \sum_{\sigma \in \mathcal{S}_N} Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \\ &\quad \times \tilde{C}_2(\lambda_{\sigma(1)}) \dots \tilde{C}_2(\lambda_{\sigma(p)}) \tilde{C}_1(\lambda_{\sigma(p+1)}) \dots \tilde{C}_1(\lambda_{\sigma(n)}) |0\rangle \\ &= \sum_{\sigma \in \mathcal{S}_N} Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \sum_{i_1 < \dots < i_p} \sum_{i_{p+1} < \dots < i_n} 2^{\frac{p(p-1)+(n-p)(n-p-1)}{2}} \\ &\quad \times \prod_{l=1}^p \prod_{k=p+1}^n a(\lambda_{\sigma(l)}, \xi_{i_k}) \det \mathcal{B}_p(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}; \xi_{i_1}, \dots, \xi_{i_p}) \\ &\quad \times \det \mathcal{B}_{n-p}(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n)}; \xi_{i_{p+1}}, \dots, \xi_{i_n}) \prod_{j=1}^{\overrightarrow{p}} E_{(i_j)}^{23} \prod_{j=p+1}^{\overrightarrow{n}} E_{(i_j)}^{13} |0\rangle \end{aligned}$$

with the sets $\{i_1, \dots, i_p\} \cap \{i_{p+1}, \dots, i_n\} = \emptyset$ and the prefactor

Y_R being

$$\begin{aligned}
& Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \\
&= \frac{1}{p!(n-p)!} c_{1\dots n}^\sigma B_{n-p}^* \left(\lambda_{p+1}^{(1)}, \dots, \lambda_n^{(1)} \mid \lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n)} \right) \\
&\times \prod_{k=p+1}^n \prod_{l=1}^p \left(-\frac{2a(\lambda_{\sigma(k)}, \lambda_{\sigma(l)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(k)})} \right).
\end{aligned}$$

Here $c_{1,\dots,n}^\sigma$ has the decomposition law $c_{1\dots n}^{\sigma'\sigma} = c_{\sigma'(1\dots n)}^\sigma c_{1\dots n}^{\sigma'}$ with $c_{1\dots n}^{\sigma_i} = c_{i\ i+1} \equiv c(v_i, v_{i+1})$ for an elementary permutation σ_i , the c -number B_p^* is given by

$$\begin{aligned}
B_p^* \left(\lambda_1^{(1)}, \dots, \lambda_p^{(1)} \mid \lambda_1, \dots, \lambda_p \right) &= \sum_{\sigma \in \mathcal{S}_p} \prod_{k=1}^p \left(-b(\lambda_k^{(1)}, \lambda_{\sigma(k)}) \right) \\
&\times \prod_{j \neq \sigma(k), \dots, \sigma(p)} \frac{c(\lambda_k^{(1)}, \lambda_j)}{2a(\lambda_{\sigma(k)}, \lambda_j)} \prod_{l=k+1}^p 2a(\lambda_k^{(1)}, \lambda_{\sigma(l)}),
\end{aligned}$$

and the elements of the $n \times n$ matrix $\mathcal{B}_n(\{\lambda_i\}; \{\xi_j\})$ are

$$(\mathcal{B}_n)_{\alpha\beta} = b(\lambda_\alpha, \xi_\beta) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_\beta)$$

and we define the notation $\prod_{i=1}^{\overrightarrow{n}} f_i \equiv f_1 \dots f_n$.

For our later use, we also introduce the notation $\prod_{i=1}^{\overleftarrow{n}} f_i \equiv f_n \dots f_1$.

By a similar procedure, one may prove the following proposition for the dual Bethe state $\langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)}) |$:

Proposition 6 *The dual Bethe state $\langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)}) |$ of the supersymmetric t - J model can be represented by*

$$\begin{aligned}
& \langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)}) | \\
&= \sum_{\sigma \in \mathcal{S}_N} Y_L(\{\mu_{\sigma(i)}\}, \{\mu_{\sigma(j)}^{(1)}\}) \langle 0 | \tilde{B}_1(\mu_{\sigma(n)}) \dots \tilde{B}_1(\mu_{\sigma(p+1)}) \\
&\quad \times \tilde{B}_2(\mu_{\sigma(p)}) \dots \tilde{B}_2(\mu_{\sigma(1)}) \\
&= \sum_{\sigma \in \mathcal{S}_N} Y_L(\{\mu_{\sigma(i)}\}, \{\mu_{\sigma(j)}^{(1)}\}) \sum_{i_1 < \dots < i_p} \sum_{i_{p+1} < \dots < i_n} \\
&\quad \times (-1)^n 2^{-\frac{p(p-1) + (n-p)(n-p-1)}{2}} \prod_{l=1}^p \prod_{k=p+1}^n a(\mu_{\sigma(l)}, \xi_{i_k}) \\
&\quad \times \prod_{l=1}^p \prod_{k \neq i_l, i_{p+1}, \dots, i_n}^N a^{-1}(\xi_k, \xi_{i_l}) \prod_{l=p+1}^n \prod_{k=1, \neq i_l}^N a^{-1}(\xi_k, \xi_{i_l}) \\
&\quad \times \det \mathcal{B}_p(\mu_{\sigma(1)}, \dots, \mu_{\sigma(p)}; \xi_{i_1}, \dots, \xi_{i_p}) \\
&\quad \times \det \mathcal{B}_{n-p}(\mu_{\sigma(p+1)}, \dots, \mu_{\sigma(n)}; \xi_{i_{p+1}}, \dots, \xi_{i_n}) \langle 0 | \prod_{j=p+1}^{\overleftarrow{n}} E_{(i_j)}^{31} \prod_{j=1}^{\overleftarrow{p}} E_{(i_j)}^{32},
\end{aligned}$$

where the prefactor Y_L is

$$Y_L(\{\mu_{\sigma(i)}\}, \{\mu_{\sigma(j)}^{(1)}\})$$

$$\begin{aligned}
&= \frac{1}{p!(n-p)!} c_{1\dots n}^\sigma B_{n-p}^{**} \left(\mu_{p+1}^{(1)}, \dots, \mu_n^{(1)} \mid \mu_{\sigma(p+1)}, \dots, \mu_{\sigma(n)} \right) \\
&\quad \times \prod_{k=p+1}^n \prod_{l=1}^p \left(-\frac{2a(\mu_{\sigma(k)}, \mu_{\sigma(l)})}{a(\mu_{\sigma(l)}, \mu_{\sigma(k)})} \right),
\end{aligned}$$

and the c -number B_p^{**} is given by

$$\begin{aligned}
&B_p^{**} \left(\mu_1^{(1)}, \dots, \mu_p^{(1)} \mid \mu_1, \dots, \mu_p \right) \\
&= \sum_{\sigma \in \mathcal{S}_p} \prod_{k=1}^p b(\mu_k^{(1)}, \mu_{\sigma(k)}) \prod_{j \neq \sigma(k), \dots, \sigma(p)} c(\mu_k^{(1)}, \mu_j) \prod_{l=k+1}^p \frac{a(\mu_k^{(1)}, \mu_{\sigma(l)})}{2a(\mu_{\sigma(l)}, \mu_{\sigma(k)})}.
\end{aligned}$$

III.2 Determinant representation of the scalar product

The scalar product of the Bethe states with a given quantum number p is defined by

$$P_n(\{\mu_i\}_{(p,n)}, \{\lambda_j\}_{(p,n)}) = \langle \Omega_N(\{\mu_j\}_{(p,n)}) | \Omega_N(\{\lambda_j\}_{(p,n)}) \rangle.$$

The invariant property of the pseudo-vacuum state under the F -transformation, i.e. $F_{1\dots N}|0\rangle = |0\rangle$ and $\langle 0|F_{1\dots N}^{-1} = \langle 0|$, implies that in the F -basis, the scalar product P_n is

$$\begin{aligned} P_n(\{\mu_i\}_{(p,n)}, \{\lambda_j\}_{(p,n)}) &= \langle \Omega_N(\{\mu_j\}_{(p,n)}) | \Omega_N(\{\lambda_j\}_{(p,n)}) \rangle \\ &= \langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)}) | \tilde{\Omega}_N(\{\lambda_j\}_{(p,n)}) \rangle \\ &= \sum_{\sigma', \sigma} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \\ &\quad \times \langle 0 | \tilde{B}_1(\mu_{\sigma'(n)}) \dots \tilde{B}_1(\mu_{\sigma'(p+1)}) \tilde{B}_2(\mu_{\sigma'(p)}) \dots \tilde{B}_2(\mu_{\sigma'(1)}) \\ &\quad \times \tilde{C}_2(\lambda_{\sigma(1)}) \dots \tilde{C}_2(\lambda_{\sigma(p)}) \tilde{C}_1(\lambda_{\sigma(p+1)}) \dots \tilde{C}_1(\lambda_{\sigma(n)}) | 0 \rangle. \end{aligned}$$

To compute the scalar product, following, we introduce the following intermediate functions

$$\begin{aligned}
& G^{(m)}(\{\lambda_k\}_{(p,n)}, u_1, \dots, \mu_m, i_{m+1}, \dots, i_n) \\
&= \left\{ \begin{array}{l} \overleftarrow{n} \quad \overleftarrow{p} \\ \langle 0 | \prod_{k=p+1}^n E_{(i_k)}^{31} \prod_{k=m+1}^p E_{(i_k)}^{32} \tilde{B}_2(\mu_m) \dots \tilde{B}_2(\mu_1) \\ \quad \times \tilde{C}_2(\lambda_1) \dots \tilde{C}_2(\lambda_p) \tilde{C}_1(\lambda_{p+1}) \dots \tilde{C}_1(\lambda_n) | 0 \rangle \\ \quad (m \leq p), \\ \\ \overleftarrow{n} \\ \langle 0 | \prod_{k=m+1}^n E_{(i_k)}^{31} \tilde{B}_1(\mu_m) \dots \tilde{B}_1(\mu_{p+1}) \tilde{B}_2(\mu_p) \tilde{B}_2(\mu_1) \\ \quad \times \tilde{C}_2(\lambda_1) \dots \tilde{C}_2(\lambda_p) \tilde{C}_1(\lambda_{p+1}) \dots \tilde{C}_1(\lambda_n) | 0 \rangle \\ \quad (m > p). \end{array} \right.
\end{aligned}$$

where the lower indices of $E_{(i_k)}^{32}$ and $E_{(i_k)}^{31}$ satisfy the relations $i_{m+1} < \dots < i_p$, $i_{p+1} < \dots < i_n$ and $\{i_1, \dots, i_p\} \cap \{i_{p+1}, \dots, i_n\} = \emptyset$.

Thus, the scalar product can be rewritten as

$$\begin{aligned}
& P_n(\{\mu_i\}_{(p,n)}, \{\lambda_j\}_{(p,n)}) \\
&= \sum_{\sigma, \sigma'} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \\
&\quad \times G^{(n)}(\{\lambda_{\sigma(j)}\}_{(p,n)}, \{\mu_{\sigma'(k)}\}_{(p,n)}).
\end{aligned}$$

Inserting a complete set, the first formula of $G^{(m)}$ becomes

$$\begin{aligned}
& G^{(m)}(\{\lambda_k\}_{(p,n)}, \mu_1, \dots, \mu_m, i_{m+1}, \dots, i_n) \\
&= \sum_{j \neq i_{m+1}, \dots, i_n}^N \langle 0 | \overleftarrow{\prod}_{k=p+1}^n E_{(i_k)}^{31} \overleftarrow{\prod}_{k=m+1}^p E_{(i_k)}^{32} \tilde{B}_2(\mu_m) \\
&\quad \times \overrightarrow{\prod}_{k=m+1}^{m+q} E_{(i_k)}^{23} E_{(j)}^{23} \overrightarrow{\prod}_{k=m+q+1}^p E_{(i_k)}^{23} \overrightarrow{\prod}_{k=p+1}^n E_{(i_k)}^{13} | 0 \rangle \\
&\quad \times G^{(m-1)}(\{\lambda_k\}_{(p,n)}, \mu_1, \dots, \mu_{m-1}, i_{m+1}, \\
&\quad \quad \dots, i_{m+q}, j, i_{m+q+1}, \dots, i_n) \quad (0 \leq q \leq p - m).
\end{aligned}$$

For the recursive relation (3), the initial function $G^{(0)}$ can be easily given by

$$\begin{aligned}
& G^{(0)}(\{\lambda_k\}_{(p,n)}, i_1, \dots, i_n) \\
&= \langle 0 | \overleftarrow{\prod}_{k=p+1}^n E_{(i_k)}^{31} \overleftarrow{\prod}_{k=1}^p E_{(i_k)}^{32} \overleftarrow{\prod}_{k=1}^p \tilde{C}_2(\lambda_k) \overleftarrow{\prod}_{k=p+1}^n \tilde{C}_1(\lambda_k) | 0 \rangle \\
&= 2^{\frac{p(p-1)+(n-p)(n-p-1)}{2}} \overleftarrow{\prod}_{l=1}^p \overleftarrow{\prod}_{k=p+1}^n a(\lambda_l, \xi_{i_k}) \\
&\quad \times \det \mathcal{B}_p(\lambda_1, \dots, \lambda_p; \xi_{i_1}, \dots, \xi_{i_p}) \\
&\quad \times \det \mathcal{B}_{n-p}(\lambda_{p+1}, \dots, \lambda_n; \xi_{i_{p+1}}, \dots, \xi_{i_n}).
\end{aligned}$$

And the prefactor of the function $G^{(m-1)}$ can also be easily obtained

by

$$\begin{aligned}
& \langle 0 | \overleftarrow{\prod}_{k=p+1}^n E_{(i_k)}^{31} \overleftarrow{\prod}_{k=m+1}^p E_{(i_k)}^{32} \tilde{B}_2(\mu_m) \\
& \quad \times \overrightarrow{\prod}_{k=m+1}^{m+q} E_{(i_k)}^{23} E_{(j)}^{23} \overrightarrow{\prod}_{k=m+q+1}^p E_{(i_k)}^{23} \overrightarrow{\prod}_{k=p+1}^n E_{(i_k)}^{13} | 0 \rangle \\
& = -(-1)^q 2^{-(p-m)} \cdot b(\mu_m, \xi_j) \prod_{l=m+1}^p a(\mu_m, \xi_{i_l}) \prod_{l=p+1}^n a(\mu_m, \xi_{i_l}) \\
& \quad \times \prod_{k \neq j, i_{p+1}, \dots, i_n}^N a^{-1}(\xi_k, \xi_j).
\end{aligned}$$

Then by using the above two relations, we arrive at our main result:

Theorem 3 *Let the spectral parameters $\{\lambda_k\}$ of the Bethe state $|\Omega_N(\{\lambda_k\}_{(p,n)})\rangle$ be solutions of the BAE. The scalar products $P_n(\{\mu_k\}_{(p,n)}, \{\lambda_k\}_{(p,n)})$ are represented by*

$$\begin{aligned}
& P_n(\{\mu_k\}_{(p,n)}, \{\lambda_k\}_{(p,n)}) \\
& = (-1)^n \sum_{\sigma, \sigma' \in \mathcal{S}_n} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \\
& \quad \times \{ \det \mathcal{F}(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}; \mu_{\sigma'(1)}, \dots, \mu_{\sigma'(p)}) \\
& \quad \quad \times \det \mathcal{N}_1(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n)}) \}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k, l_k, \varrho_{m_k}^k} ' \sum_{\substack{j_1, \dots, j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(\sigma'(j_1) \dots \sigma'(j_p))} \\
& \quad \times (A_2)_{\sigma(1) \sigma'(j_1)} \cdots (A_2)_{\sigma(p) \sigma'(j_p)} \\
& \quad \times \det \mathcal{N}_2(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n)}) \\
& + \sum_{k, l_k, \rho_{m_k}^k} '' \det A_3(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}; \mu_{\sigma'(1)}, \dots, \mu_{\sigma'(p)}) \\
& \quad \times \det \mathcal{N}_3(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n)}) \\
& + \sum_{k, l_k, \varrho_{m_k}^k} ' \sum_{k', l'_{k'}, \rho_{m'_{k'}}^{k'}} '' \sum_{\substack{j_1, \dots, j_p=1 \\ j_1 \neq \dots \neq j_p}}^p (-1)^{\tau(\sigma'(j_1) \dots \sigma'(j_p))} \\
& \quad \times (A_4)_{\sigma(1) \sigma'(j_1)} \cdots (A_2)_{\sigma(p) \sigma'(j_p)} \\
& \quad \times \det \mathcal{N}_4(\lambda_{\sigma(p+1)}, \dots, \lambda_{\sigma(n)}; \mu_{\sigma'(p+1)}, \dots, \mu_{\sigma'(n)}) \}.
\end{aligned}$$

For expressions of \mathcal{F} , A_2 , A_3 , A_4 , $\mathcal{N}_1, \dots, \mathcal{N}_4$, One may find in our paper: “*Commun. Math. Phys.* **268** (2006) 505-541”.

Remark: On the other hand, if we compute the scalar product by starting from the dual state $\langle \Omega_N(\{\lambda_j\}_{(p,n)}) |$, then by using the same procedure, we have the scalar product $P_n^L(\{\lambda_k\}_{(p,n)}, \{\mu_j\}_{(p,n)})$ is given by

$$\begin{aligned}
& P_n^L(\{\lambda_k\}_{(p,n)}, \{\mu_j\}_{(p,n)}) \\
& = \sum_{\sigma, \sigma' \in \mathcal{S}_n} Y_L(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) Y_R(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) \\
& \quad \times G^{(n)}(\{\mu_{\sigma'(j)}\}_{(p,n)}, \{\lambda_{\sigma(k)}\}_{(p,n)}).
\end{aligned}$$

III.3 Determinant representations of correlation functions

A k -point correlation function is defined by

$$F_n^{\epsilon^1, \dots, \epsilon^k} = \langle \Omega_N(\{\mu_j\}) | \epsilon_{i_1}^1 \dots \epsilon_{i_k}^k | \Omega_N(\{\lambda_j\}) \rangle,$$

• Fermion representations of the local operators

Before we compute the correlation functions, we first introduce the proposition

Proposition 7 (*Göhmann-Korepin*) *The local spin and field operators of the supersymmetric t - J models can be represented in terms of monodromy matrix as follows.*

$$\begin{aligned} (1 - n_{\kappa, \uparrow}) c_{\kappa, \downarrow} &= \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot B_1(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad , \\ (1 - n_{\kappa, \downarrow}) c_{\kappa, \uparrow} &= \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot B_2(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad , \\ (1 - n_{\kappa, \uparrow}) c_{\kappa, \downarrow}^\dagger &= \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot C_1(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad , \\ (1 - n_{\kappa, \downarrow}) c_{\kappa, \uparrow}^\dagger &= \prod_{j=1}^{n-1} t(\xi_j) \cdot C_2(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad , \\ (1 - n_{\kappa, \downarrow})(1 - n_{\kappa, \uparrow}) &= \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot D(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad , \end{aligned}$$

$$\begin{aligned}
S_\kappa^\dagger &= - \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot A_{21}(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad , \\
S_\kappa &= - \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot A_{12}(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad , \\
S_\kappa^z &= - \frac{1}{2} \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot (A_{11}(\xi_\kappa) - A_{22}(\xi_\kappa)) \cdot \prod_{j=\kappa+1}^N t(\xi_j) \quad .
\end{aligned}$$

Here $n_{\kappa\downarrow}n_{\kappa\uparrow} = 0$ since the double occupancy of lattice sites on the restricted Hilbert space of the supersymmetric t - J model is excluded.

• Correlation functions

In general, with the help of the representations for the local operators, we may compute any correlation functions. Here we only give two examples.

i) One-point function

For the local fermion operator $(1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}$, the correlation functions are given by

$$\begin{aligned}
&F_n^\dagger(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p+1,n+1)}) \\
&= \langle \Omega_N(\{\mu_j\}_{(p,n)}) | \cdot (1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow} \cdot | \Omega_N(\{\lambda_j\}_{(p+1,n+1)}) \rangle \\
&= \sum_{\sigma \in \mathcal{S}_{n+1}} \sum_{\sigma' \in \mathcal{S}_n} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \\
&\quad \times \langle 0 | \prod_{i=p+1}^{\overleftarrow{n}} B_1(\mu_{\sigma'(i)}) \prod_{i=1}^{\overleftarrow{p}} B_2(\mu_{\sigma'(i)}) \cdot (1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger \cdot
\end{aligned}$$

$$\times \prod_{i=1}^{\overrightarrow{p+1}} C_2(\lambda_{\sigma(i)}) \prod_{i=p+2}^{\overrightarrow{n+1}} C_1(\lambda_{\sigma(i)}) |0\rangle,$$

where $\{\mu_j\}, \{\lambda_k\}$ are solutions of BAE, p and $p+1$ are quantum numbers of the corresponding states. For the representations of the correlation functions, we prove the following proposition:

Proposition 8 *If both the Bethe state $|\Omega_N(\{u_j\})\rangle$ and the dual Bethe state $\langle\Omega_N(\{u_j\})|$ ($u_j = \lambda_j, \mu_j$) are eigenstates of the transfer matrix, then the correlation functions corresponding to the local fermion operators $(1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}$ can be represented by*

$$\begin{aligned} & F_n^\uparrow(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p+1,n+1)}) \\ &= (-1)^{n+1} \sum_{\sigma \in \mathcal{S}_{n+1}} \sum_{\sigma' \in \mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\}) \phi_\kappa^{-1}(\{\lambda_k\}) \\ & \quad \times P_{n+1}(\xi_\kappa, \mu_{\sigma'(1)}, \dots, \mu_{\sigma'(n)}; \{\lambda_{\sigma(j)}\}_{(p+1,n+1)}), \end{aligned}$$

respectively, where $\phi_i(\{\mu_j\}) = \prod_{k=1}^i \prod_{l=1}^n a^{-1}(\mu_l, \xi_k)$.

Proof. From the definition of F_n^\uparrow , we have

$$\begin{aligned} & F_{n+1}^\uparrow(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p+1,n+1)}) \\ &= \langle\Omega_N(\{\mu_j\}_{(p,n)})| \cdot (1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow} \cdot |\Omega_N(\{\lambda_j\}_{(p+1,n+1)})\rangle \\ &= \langle\Omega_N(\{\mu_j\}_{(p,n)})| \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot B_2(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) |\Omega_N(\{\lambda_j\}_{(p+1,n+1)})\rangle \end{aligned}$$

$$\begin{aligned}
&= (-1)^{n+1} \sum_{\sigma \in \mathcal{S}_{n+1}} \sum_{\sigma' \in \mathcal{S}_n} \prod_{j=1}^n \prod_{k=1}^{\kappa-1} a^{-1}(\mu_j, \xi_k) \prod_{j=1}^{n+1} \prod_{k=\kappa+1}^N a^{-1}(\lambda_j, \xi_k) \\
&\quad \times P_{n+1} \left(\xi_\kappa, \mu_{\sigma'(1)}, \dots, \mu_{\sigma'(n)}; \{\lambda_{\sigma(j)}\}_{(p+1, n+1)} \right).
\end{aligned}$$

Then by using the relation

$$\prod_{j=1}^n \prod_{k=1}^N a^{-1}(\lambda_j, \xi_k) = 1,$$

which is from the BAE and the NBAE. □

ii) two-point function

For two-point case, we consider the correlation function for two adjacent fermion operators $(1 - n_{\kappa, \downarrow})c_{\kappa, \uparrow}^\dagger$ and $(1 - n_{\kappa+1, \downarrow})c_{\kappa+1, \uparrow}$.

With the help of the representations of the fermion operators, the correlation function is defined by

$$\begin{aligned}
&F_n^{\uparrow \dagger, \uparrow}(\{\mu_j\}_{(p, n)}, \xi_\kappa, \xi_{\kappa+1}, \{\lambda_k\}_{(p, n)}) \\
&= \langle \Omega_N^p(\{\mu_j\}) | (1 - n_{\kappa, \downarrow})c_{\kappa, \uparrow}^\dagger (1 - n_{\kappa+1, \downarrow})c_{\kappa+1, \uparrow} | \Omega_N^p(\{\lambda_k\}) \rangle \\
&= \langle \Omega_N^p(\{\mu_j\}) | \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot C_2(\xi_\kappa) B_2(\xi_{\kappa+1}) \cdot \prod_{j=\kappa+2}^N t(\xi_j) | \Omega_N^p(\{\lambda_k\}) \rangle.
\end{aligned}$$

Here we have used the following property: for the supersymmetric t - J model with periodic boundary condition, the transfer matrices satisfy the relation $\prod_{i=1}^N t(\lambda_i) = 1$.

Then with the help of the commutation relations

$$\begin{aligned}
A_{ab}(\lambda)C_c(\mu) &= \frac{r(\lambda - \mu)_{de}^{bc}}{a(\lambda - \mu)}C_e(\mu)A_{ad}(\lambda) + \frac{b(\lambda - \mu)}{a(\lambda - \mu)}C_b(\mu)A_{ac}(\lambda), \\
D(\lambda)C_c(\mu) &= \frac{1}{a(\mu, \lambda)}C_c(\mu)D(\lambda) - \frac{b(\mu, \lambda)}{a(\mu, \lambda)}C_c(\lambda)D(\mu), \\
B_a(\lambda)C_b(\mu) &= -C_b(\mu)B_a(\lambda) \\
&\quad + \frac{b(\lambda, \mu)}{a(\lambda, \mu)}[D(\mu)A_{ab}(\lambda) - D(\lambda)A_{ab}(\mu)], \\
C_a(\lambda)C_b(\mu) &= -\frac{1}{a(\mu, \lambda)}C_b(\mu)C_a(\lambda) + \frac{b(\mu, \lambda)}{a(\mu, \lambda)}C_b(\lambda)C_a(\mu),
\end{aligned}$$

one proves the following proposition

Proposition 9 *If both the Bethe state $|\Omega_N(\{u_j\})\rangle$ and the dual Bethe state $\langle\Omega_N(\{u_j\})|$ ($u_j = \lambda_j, \mu_j$) are eigenstates of the transfer matrix, then the two-point correlation functions associated with the local fermion operators $(1 - n_{\kappa, \downarrow})c_{\kappa, \uparrow}^\dagger$ and $(1 - n_{\kappa+1, \downarrow})c_{\kappa+1, \uparrow}$ can be represented by*

$$\begin{aligned}
&F_n^{\uparrow\uparrow, \uparrow}(\{\mu_j\}_{(p, n)}, \xi_\kappa, \xi_{\kappa+1}, \{\lambda_k\}_{(p, n)}) \\
&= \sum_{\sigma \in \mathcal{S}_n} \sum_{\sigma' \in \mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\}) \phi_{\kappa+1}^{-1}(\{\lambda_k\}) \sum_{i=1}^p (-1)^{i-1} \frac{b(\xi_{\kappa+1}, \lambda_{\sigma(i)})}{a(\xi_{\kappa+1}, \lambda_{\sigma(i)})} \\
&\quad \times \left\{ \sum_{j=i+1}^p \frac{b(\xi_{\kappa+1}, \lambda_{\sigma(j)})}{a(\xi_{\kappa+1}, \lambda_{\sigma(j)})} \prod_{l=i+1}^{j-1} \frac{c(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(l)}, \xi_\kappa)} \right. \\
&\quad \times \left. \prod_{k=i+1, \neq j}^p \frac{c(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})} \prod_{\alpha=1}^N a(\lambda_{\sigma(j)}, \xi_\alpha) \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \mathcal{P} \left(i + 1; \lambda_{\sigma(i)}; \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda'_{\sigma(f)}\}_{(p,n)} \right) \\
& - \prod_{j=i+1}^p \frac{c(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})} \prod_{\alpha=1}^N a(\lambda_{\sigma(i)}, \xi_{\alpha}) \\
& \times \mathcal{P} \left(i + 1; \xi_{\kappa+1}; \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda^*_{\sigma(f)}\}_{(p,n)} \right) \\
& - \sum_{j=i+1}^p \frac{b(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})} \prod_{l=i+1}^{j-1} \frac{c(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \\
& \times \prod_{k=i+1, \neq j}^p \frac{c(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})} \prod_{\alpha=1}^N a(\lambda_{\sigma(j)}, \xi_{\alpha}) \\
& \times \mathcal{P} \left(i + 1; \xi_{\kappa+1}; \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda''_{\sigma(f)}\}_{(p,n)} \right) \Big\},
\end{aligned}$$

where \mathcal{P} is given by

$$\begin{aligned}
& \mathcal{P}(e; \delta; \{\mu_{\sigma'(j)}\}; \{\lambda_{\sigma(k)}\}) \\
& = \prod_{i=e}^n \frac{1}{a(\lambda_{\sigma(i)}, \delta)} P_n^L \left(\{\mu_{\sigma'(j)}\}_{(p,n)}; \{\lambda_{\sigma(k)}\}_{(p,n)} \right) \\
& - \sum_{j=p+1}^n \frac{b(\lambda_{\sigma(j)}, \delta)}{a(\lambda_{\sigma(j)}, \delta)} \prod_{k=p+1}^{j-1} \frac{c(\lambda_{\sigma(k)}, \lambda_{\sigma(j)})}{c(\lambda_{\sigma(k)}, \delta)} \prod_{l=e, \neq j}^n \frac{1}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})} \\
& \times P_n^L \left(\{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(e)}, \dots, \lambda_{\sigma(j-1)}, \right. \\
& \quad \left. \delta, \lambda_{\sigma(j+1)}, \dots, \lambda_{\sigma(n)}\}_{(p,n)} \right) \\
& - \sum_{i=e}^p \frac{b(\lambda_{\sigma(i)}, \delta)}{a(\lambda_{\sigma(i)}, \delta)} \prod_{k=1}^{i-1} \frac{c(\lambda_{\sigma(k)}, \lambda_{\sigma(i)})}{c(\lambda_{\sigma(k)}, \delta)} \prod_{j=1, \neq i}^p \frac{1}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(i)})} \\
& \times \left[\prod_{l=p+1}^n \frac{1}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} P_n^L \left(\{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda_{\sigma(1)}, \dots, \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \lambda_{\sigma(e), \dots, \lambda_{\sigma(i-1)}, \delta, \lambda_{\sigma(i+1)}, \dots, \lambda_{\sigma(n)} \}_{(p,n)} \\
& - \sum_{l=p+1}^n \frac{b(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \prod_{m=p+1}^{l-1} \frac{c(\lambda_{\sigma(m)}, \lambda_{\sigma(l)})}{c(\lambda_{\sigma(m)}, \lambda_{\sigma(i)})} \\
& \times \prod_{q=p+1, \neq l}^n \frac{1}{a(\lambda_{\sigma(q)}, \lambda_{\sigma(l)})} \\
& \times P_n^L \left(\{ \mu_{\sigma'(d)} \}_{(p,n)}; \{ \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(e)}, \dots, \lambda_{\sigma(i-1)}, \delta, \right. \\
& \left. \lambda_{\sigma(i+1)}, \dots, \lambda_{\sigma(l-1)}, \lambda_{\sigma(i)}, \lambda_{\sigma(l+1)}, \dots, \lambda_{\sigma(n)} \}_{(p,n)} \right) \Big]
\end{aligned}$$

and the spectral parameters λ' , λ^* and λ'' are given by

$$\begin{aligned}
\lambda'_{\sigma(k)} &= \begin{cases} \xi_{\kappa} & (k=1) \\ \lambda_{\sigma(k-1)} & (2 \leq k \leq i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n \text{ and } k \neq j) \\ \xi_{\kappa+1} & (k=j) \end{cases}, \\
\lambda^*_{\sigma(k)} &= \begin{cases} \xi_{\kappa} & (k=1) \\ \lambda_{\sigma(k-1)} & (k=2, \dots, i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n) \end{cases}, \\
\lambda''_{\sigma(k)} &= \begin{cases} \xi_{\kappa} & (k=1) \\ \lambda_{\sigma(k-1)} & (2 \leq k \leq i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n \text{ and } k \neq j) \\ \lambda_{\sigma(i)} & (k=j) \end{cases},
\end{aligned}$$

respectively.

Part IV: Conclusions and Discussions

In this work, we constructed the Drinfeld twists for the supersymmetric fermion models. Working in the F -basis, we obtained determinant representations of correlation functions for the supersymmetric t - J model. The results make us possible to study the following interesting problems:

- i). To discuss the thermodynamical properties of the correlation functions.
- ii). To study the time-and temperature-correlation functions of the models.
- iii). To study the relevant mathematical problems, such as the alternative sign matrix.

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