

- Definition of Operator  $\mathbf{c}(\zeta, \alpha)$   
Singular part of the partial fraction

$$\mathbf{c}(\zeta, \alpha) = q^{\alpha - \mathbb{S}} \text{sing } \mathbf{k}^+(\zeta, \alpha)$$

where

$$\mathbf{k}^+(\zeta, \alpha)(X) = (1 - q^{2(\alpha - \mathbb{S})}) \zeta^{\alpha - \mathbb{S}} \times \\ \times \text{Tr}_{A,a}^+ \left( q^{2\alpha D_A} \sigma_a^+ T_{\{A,a\}}(\zeta)^{-1} X T_{\{A,a\}}(\zeta) \right),$$

$$T_{\{A,a\}}^+(\zeta) = L_{\{A,a\},n}^+(\zeta/\xi_n) \cdots L_{\{A,a\},1}^+(\zeta/\xi_1).$$

- Triangular  $L$  operator

$$L_{\{A,a\},j}^+(\zeta) = \begin{pmatrix} A_j & 0 \\ C_j & D_j \end{pmatrix}_a,$$

$$A_j = L_{A,j}^+(q^{-1}\zeta/\xi_j) q^{-\sigma_j^3/2},$$

$$C_j = \gamma(\zeta/\xi_j) L_{A,j}^+(q\zeta/\xi_j) \sigma_j^+ q^{-2D_A + \frac{1}{2}},$$

$$D_j = \beta(\zeta/\xi_j) L_{A,j}^+(q\zeta/\xi_j) q^{\sigma_j^3/2}.$$

• Reduction property

Using  $[2D_A - \sigma_a^3 - \sigma_1^3, L_{\{A,a\},1}^+(\zeta)] = 0$  and  $T_{\{A,a\}}(\zeta) = T_2(\zeta)L_{\{A,a\},1}^+(\zeta/\xi_1)$  we get

$$\begin{aligned}
& \text{Tr}_{A,a}^+(\dots) \\
&= q^{(\alpha-1)(\sigma_1^3-1)} \left\{ \text{Tr}_{A,a}^+ \sigma_a^+ q^{2D_A} \times \right. \\
& \quad \times L_{\{A,a\},1}^+(\zeta/\xi_1)^{-1} q^{(\alpha-1)(2D_A - \sigma_a^3 - \sigma_1^3)} q^{\alpha\sigma_1^3} \times \\
& \quad \left. \times T_2(\zeta)^{-1} X_2 T_2(\zeta) L_{\{A,a\},1}^+(\zeta/\xi_1) \right\} \\
&= q^{(\alpha-1)(\sigma_1^3-1)} \text{Tr}_{A,a}^+ \left( \{L_{\{A,a\},1}^+(\zeta/\xi_1)\}^{t_1} \times \right. \\
& \quad \times \sigma_a^+ q^{2D_A} q^{\sigma_1^3} \{L_{\{A,a\},1}^+(\zeta/\xi_1)^{-1}\}^{t_1} \left. \right)^{t_1} \times \\
& \quad \times q^{(\alpha-1)(2D_A - \sigma_a^3)} T_2(\zeta)^{-1} X_2 T_2(\zeta).
\end{aligned}$$

We can compute

$$\begin{aligned}
& \left( \dots \right)^{t_1} = q^{2D_A} \sigma_a^+ + \\
& + \frac{\gamma(q\zeta/\xi_1)}{\beta(q\zeta/\xi_1)} \sigma_1^+ \tau_a^- - \frac{\gamma(q^{-1}\zeta/\xi_1)}{\beta(q^{-1}\zeta/\xi_1)} \sigma_1^+ \tau_a^+.
\end{aligned}$$

and reach

$$\begin{aligned} & \mathbf{k}_{[1,n]}^+(\zeta, \alpha)(q^{\alpha\sigma_1^3} X_{[2,n]}) \\ &= q^{(\alpha-1)\sigma_1^3} \mathbf{k}_{[2,n]}^+(\zeta, \alpha)(X_{[2,n]}) + \\ &+ \sigma_1^+ \Delta_q \left( \frac{\gamma(\zeta/\xi_1)}{\beta(\zeta/\xi_1)} \mathbf{Q}_{[2,n]}^+(\zeta, \alpha-1)(X_{[2,n]}) \right). \end{aligned}$$

• Annihilation property

$$\begin{aligned} \mathbf{c}_n^+(\alpha)(X) &= \frac{(1 - q^{2(\alpha-S)})(q\xi_n)^{\alpha-S}}{\prod_{j=1}^{n-1} \beta(\xi_n/\xi_j)} \times \\ &\times \tilde{\mathbf{Q}}^+(\xi_n, \alpha) \left( (\mathbf{a}, 1)_n X \begin{pmatrix} 1 \\ -\mathbf{a} \end{pmatrix}_n \right) \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathbf{Q}}^+(\zeta, \alpha)(\tilde{X}) &= q^{-S_{[1,n-1]}} \times \\ &\times \text{Tr}_A^\pm \left( q^{2\alpha D_A} T_{A,[1,n-1]}^+(q\zeta)^{-1} \tilde{X} T_{A,[1,n-1]}^+(\zeta/q) \right) \\ &\times q^{-S_{[1,n-1]}} \end{aligned}$$

• Grassmann relation

$$\begin{aligned} & \mathbf{k}^+(\zeta_1, \alpha-1) \mathbf{k}^+(\zeta_2, \alpha) + \mathbf{k}^+(\zeta_2, \alpha-1) \mathbf{k}^+(\zeta_1, \alpha) \\ &= \Delta_{q,\zeta_1} F(\zeta_1, \zeta_2) + \Delta_{q,\zeta_2} F(\zeta_2, \zeta_1), \end{aligned}$$

- Jordan Wigner transformation

$$\psi_j^\pm = \sigma_j^\pm i^{\mp} \sum_{l=1}^{j-1} \sigma_l^3 \in \mathcal{V}_{[1,n]}.$$

- Adjoint action

$$\Psi_j^\pm : \mathcal{V}_\alpha^{(s)} \rightarrow \mathcal{V}_{\alpha \mp 1}^{(s \pm 1)}, \quad \Phi_j^\pm(\alpha) : \mathcal{V}_{\alpha \pm 1}^{(s)} \rightarrow \mathcal{V}_\alpha^{(s \pm 1)}$$

$$\Psi_j^\pm(1) = 0, \quad \Phi_j^\pm(\alpha)(q^{(\alpha \pm 1)} \sigma^3) = 0.$$

$$\Psi_j^\pm(X) = \psi_j^\pm X - (-1)^s X \psi_j^\pm$$

$$\Phi_j^\pm(X) = \frac{1}{1 + i^{\mp 2} (\alpha - s)} (\psi_j^\pm X + i^{\mp 2 \alpha} X \psi_j^\pm)$$

- Invariant fermions

$$\widehat{\Psi}^\pm(\zeta) = \sum_{j=1}^n \Psi_j^\pm \frac{i^j \gamma(\zeta / \xi_j)}{\prod_{l=1}^j \beta(\zeta / \xi_l)},$$

$$\widehat{\Phi}^\pm(\zeta) = \sum_{j=1}^n \Phi_j^\pm \frac{i^j \gamma(\zeta / \xi_j)}{\prod_{l=1}^j \beta(\zeta / \xi_l)}.$$

- Annihilation and creation operators

$$\widehat{\Psi}_k^\pm = -\text{res}_{\zeta=\xi_k} \widehat{\Psi}^\pm(\zeta) \frac{d\zeta}{\zeta},$$

$$\widehat{\Phi}_k^\pm = -i \widehat{\Phi}^\pm(i\xi_k).$$

$$\widehat{\Psi}_k^\pm = \prod_{l=1}^{k-1} \frac{1 + \xi_k^2/\xi_l^2}{1 - \xi_k^2/\xi_l^2} \sum_{j=1}^n \Psi_j^\pm P_{kj},$$

$$\widehat{\Phi}_k^\pm = \prod_{l=1}^{k-1} \frac{1 - \xi_k^2/\xi_l^2}{1 + \xi_k^2/\xi_l^2} \sum_{j=1}^n \Phi_j^\pm (P^{-1})_{jk}.$$

$$P_{jk} = \begin{cases} \frac{2\xi_j/\xi_k}{1 - \xi_j^2/\xi_k^2} \prod_{l=j+1}^{k-1} \frac{1 + \xi_j^2/\xi_l^2}{1 - \xi_j^2/\xi_l^2} & \text{if } j < k; \\ 1 & \text{if } j = k; \\ 0 & \text{if } j > k. \end{cases}$$

$$P_{jk}^{-1} = \begin{cases} \frac{2\xi_k/\xi_j}{1 - \xi_k^2/\xi_j^2} \prod_{l=j+1}^{k-1} \frac{1 + \xi_k^2/\xi_l^2}{1 - \xi_k^2/\xi_l^2} & \text{if } j < k; \\ 1 & \text{if } j = k; \\ 0 & \text{if } j > k. \end{cases}$$

- Another set of free fermions

$$U_k^\pm = \prod_{l \neq k} \frac{(1 - \xi_k^2/\xi_l^2)^{N_l^\pm}}{(1 + \xi_k^2/\xi_l^2)^{N_l^\mp}},$$

$$\chi_k^+ = \widehat{\Psi}_k^+ U_k^+ \xi_k^{\alpha - S - 1} i^\alpha,$$

$$\chi_k^- = \widehat{\Psi}_k^- U_k^- \xi_k^{-\alpha + S - 1} (-1)^{N_k^-} \frac{i^{-S+1}}{1 + i^{2(\alpha - S)}},$$

$$\begin{aligned} \chi_k^{*+} &= i^{S-1} (1 + i^{2(\alpha - S)}) (-1)^{N_k^-} \xi_k^{\alpha - S + 1} \times \\ &\quad \times (U_k^-)^{-1} \widehat{\Phi}_k^+, \end{aligned}$$

$$\chi_k^{*-} = i^{-\alpha} \xi_k^{-\alpha + S + 1} (U_k^+)^{-1} \widehat{\Phi}_k^-.$$

- Cubic formula

$$\mathbf{c}_k^\pm = \chi_k^\pm \chi_k^\mp \chi_k^{*\pm} = \chi_k^\pm (1 - N_k^\pm),$$

$$N_k^\pm = \widehat{\Phi}_k^\pm \widehat{\Psi}_k^\mp = \chi_k^{*\pm} \chi_k^\mp$$